RESCEU-5/15

Multi-field formulation of gravitational particle production after inflation

Yuki Watanabe*
Research Center for the Early Universe, University of Tokyo, Tokyo 113-0033, Japan and Department of Physics, National Institute of Technology, Gunma College, Gunma 371-8530, Japan

Jonathan White†
Research Center for the Early Universe, University of Tokyo, Tokyo 113-0033, Japan and Theory Center, KEK, Tsukuba 305-0801, Japan

We study multi-field inflation models that contain a non-trivial field-space metric and a non-minimal coupling between the gravity and inflaton sectors. In such models it is known that even in the absence of explicit interaction terms the inflaton sector can decay into matter as a result of its non-minimal coupling to gravity, thereby reheating the Universe gravitationally. Using the Bogoliubov approach we evaluate the gravitational decay rates of the inflaton fields into both scalars and fermions, and analyse the reheating dynamics. We also discuss how the interpretation of the reheating dynamics differs in the so-called Jordan and Einstein frames, highlighting that the calculation of the Bogoliubov coefficients is independent of the frame in which one starts.

arXiv:1503.08430v2 [astro-ph.CO] 21 Aug 2015

I. INTRODUCTION

An epoch of inflation in the very early Universe is now firmly supported by recent observations of the cosmic microwave background (CMB) [1–4]. These observations suggest that the primordial curvature perturbation generated during inflation is nearly Gaussian and adiabatic, with a power spectrum that deviates from scale-invariance at the 5σ-level. They also suggest that the amplitude of tensor modes is relatively small, i.e. that the energy scale of inflation is low.

With inflation widely accepted as a key part of the standard model of cosmology, the question now turns to determining the exact nature of inflation and how it might be embedded in some fundamental high-energy-physics theory. Inflation models containing non-minimal gravitational coupling comprise one class of promising models. As well as being theoretically well motivated in the context of high-energy-physics theories such as string theory, see e.g. [5], their predictions also lie at the sweet spot of current observational constraints [6]. Examples include Starobinsky's original R^2 inflation (written in its scalar-tensor form) [7], Higgs inflation [8] and a whole class of so-called conformal inflation models recently proposed by Kallosh *et al.* [9]. Whilst most of these models are studied as single-field models, the high-energy-physics theories that motivate them generically predict the presence of multiple fields during inflation. As such, it is important to determine any possible signatures of multi-field effects in models with non-minimal coupling [10–15].

It has recently been demonstrated that in constraining specific models of inflation with current CMB data, details of the reheating process must be properly taken into account, even for single-field models [16–19]. This is testimony to the precision of current CMB data. Moreover, in the context of multi-field models of inflation, the primordial curvature perturbation may continue to evolve during reheating, and this evolution must therefore be tracked until an adiabatic limit is reached [20–23]. It is known that reheating can take place gravitationally in models with non-minimal gravitational coupling; even if there are no explicit interaction terms between the inflaton sector and matter, gravitational particle production takes place as a result of the non-minimal gravitational coupling [7, 24–34]. In light of the renewed interest in this class of models, in this paper we revisit the theory of gravitational reheating after inflation and present a multi-field formulation of gravitational particle production. Whilst we focus on perturbative gravitational reheating, we nevertheless employ the method of Bogoliubov transformations to determine the decay rates. As such, many of the features we discuss should also carry over to the case of non-perturbative preheating. Of course, the Bogoliubov approach recovers the standard perturbative quantum-field-theory (QFT) results when the appropriate limits are taken.

Throughout the analysis we try to pay particular attention to how the interpretation of the reheating dynamics differs in the so-called Jordan and Einstein frames. During the oscillatory phase at the end of inflation, the Hubble rate in the original Jordan frame contains an oscillatory component, and it is this

* watanabe'at'resceu.s.u-tokyo.ac.jp
† jwhite'at'post.kek.jp

oscillatory component that gives rise to particle production even in the absence of direct couplings between the inflaton sector and the decay products. The evolution of the scale factor in the Einstein frame, on the other hand, is equivalent to that of a matter-dominated universe, and can essentially be neglected. As such, the leading-order contribution to the gravitational particle production is not a result of the oscillatory nature of the Hubble rate. In its place, however, one obtains explicit gravitationally induced interaction terms between the inflaton sector and ordinary matter, through which reheating proceeds. Although the interpretation in the two frames is different, we nevertheless find that the calculation of the Bogoliubov coefficients is independent of the frame in which we start; working in conformal coordinates and requiring that the mode functions under consideration be canonically normalised leads us to a common set of variables and form of action.

A technical complication that arises in the context of multi-field models with non-minimal coupling is that even if one starts with a flat field space in the Jordan frame – i.e. a canonical, diagonal kinetic term – then one obtains a non-flat field space in the Einstein frame, where evaluation of the inflaton dynamics is simpler [35]. We thus find it necessary to work in the mass eigen-basis as defined with respect to the Einstein frame potential. As a result, if either of the Jordan frame field-space metric or non-minimal coupling are functions of some light spectator field, we find that the gravitational decay rates generically become modulated, giving rise to a modulated-reheating scenario [36–39].

The rest of this paper is organised as follows: In Sec. II we begin by outlining the class of models under consideration and by reviewing some of their key characteristics, with the review extending into Appendix A. In Sec. III we then analyse the reheating dynamics. We start, in Sec. III A, by looking at the background dynamics of the oscillating inflaton fields at the end of inflation, and in Sec. III B and Sec. III C we turn to the reheating process itself, presenting the details of the Bogoliubov calculation used to determine the decay rates. Additional details regarding the calculation of fermion production rates are included in Appendix B. Finally, Sec. IV is devoted to summary and conclusions.

II. MULTI-FIELD MODELS WITH NON-MINIMAL COUPLING

In this section we define more explicitly the class of models under consideration and also discuss the relation between formulations made in the Jordan and Einstein frames.

A. Actions in the Jordan and Einstein frames

The general class of models that we are considering take an action of the form

$$S = \int d^4x\sqrt{-g}\left\{\frac{f(\phi)R}{2} - \frac{1}{2}h_{ab}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b - V(\phi)\right\} + S_m, \tag{1}$$

where $a, b = 1...n$ label n scalar fields that are potentially all non-minimally coupled to the Ricci scalar R through the function $f(\boldsymbol{\phi})$. h_{ab} defines a non-flat field-space metric and V is some general potential depending on all the fields. We take the matter part of the action to consist of bosons and fermions, namely

$$S_m = \sum_i S_{\chi_i} + \sum_i S_{\psi_i}, \qquad \text{where} \qquad \begin{aligned} S_{\chi_i} &= \textstyle\int d^4x\sqrt{-g}\left\{-\frac{1}{2}g^{\mu\nu}\partial_\mu\chi_i\partial_\nu\chi_i - U(\chi_i)\right\}, \\ S_{\psi_i} &= -\textstyle\int d^4x\sqrt{-g}\left\{\overline{\psi}_i \overleftrightarrow{\not{D}}\psi_i + m_{\psi_i}\overline{\psi}_i\psi_i\right\}. \end{aligned} \tag{2}$$

Here $\not{D}$ is given as $\not{D} = \gamma^\mu(x)D_\mu$, with $D_\mu = \partial_\mu + \Gamma_\mu$ and $\gamma^\mu(x) = e^\mu_\alpha\gamma^\alpha$, where e^μ_α is the tetrad defining local Lorentzian coordinates, γ^α are the standard Dirac matrices satisfying[1] $\left\{\gamma^\alpha, \gamma^\beta\right\} = 2\eta^{\alpha\beta}$, and the spinor connection Γ_μ is defined as $\Gamma_\mu = (1/2)\Sigma^{\alpha\beta}e^\lambda_\alpha\nabla_\mu e_{\beta\lambda}$, where $\Sigma^{\alpha\beta} = \frac{1}{4}[\gamma^\alpha, \gamma^\beta]$. We also have $\overline{\psi}_i = \psi_i^\dagger\beta$, where $\beta = i\gamma^0$. We have omitted the conformally invariant gauge fields, as they do not play an important role in the perturbative reheating considered in this paper. See, however, [31] for a discussion on the gauge trace anomaly and its importance in the reheating process.

---

[1] Note that we are working with the signature $(-+++)$.

Matter in the above action is minimally coupled to gravity, and this "frame" is referred to as the Jordan frame. However, on making the conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, with $\Omega^2 = f(\boldsymbol{\phi})/M_{\rm Pl}^2$, the action can be re-written as

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{M_{\rm Pl}^2\tilde{R}}{2} - \frac{1}{2}S_{ab}\tilde{g}^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b - \tilde{V}\right\} + S_{\tilde{m}}, \tag{3}$$

where[2]

$$S_{ab} = \frac{M_{\rm Pl}^2}{f}\left(h_{ab} + \frac{3f_af_b}{2f}\right), \qquad \tilde{V} = \frac{M_{\rm Pl}^4 V}{f^2} \tag{4}$$

and the matter actions now take the form

$$S_{\tilde{\chi}_i} = \int d^4x\sqrt{-\tilde{g}}\left\{-\frac{1}{2}\tilde{g}^{\mu\nu}\mathcal{D}_\mu\tilde{\chi}_i\mathcal{D}_\nu\tilde{\chi}_i - \frac{U(\chi_i)}{\Omega^4}\right\}, \tag{5}$$

$$S_{\tilde{\psi}_i} = -\int d^4x\sqrt{-\tilde{g}}\left\{\bar{\tilde{\psi}}_i\overleftrightarrow{\tilde{\not{D}}}\tilde{\psi}_i + \frac{m_\psi}{\Omega}\bar{\tilde{\psi}}_i\tilde{\psi}_i\right\}. \tag{6}$$

Here we have defined

$$\tilde{\chi}_i = \frac{\chi_i}{\Omega}, \quad \tilde{\psi}_i = \Omega^{-3/2}\psi_i, \quad \mathcal{D}_\mu = \partial_\mu + \tilde{\chi}_i\partial_\mu(\ln\Omega) \quad \text{and} \quad \tilde{\not{D}} = \tilde{e}^\mu_\alpha\gamma^\alpha\left(\partial_\mu + \Gamma_\mu\right), \tag{7}$$

where $\tilde{e}^\mu_\alpha = e^\mu_\alpha/\Omega$, which gives $\tilde{\gamma}^\mu(x) = \gamma^\mu(x)/\Omega$, and the spinor connection Γ_μ is conformally invariant (see, *e.g.*, footnote 4 of [31]). In this form, the fields ϕ^a are minimally coupled to gravity and the gravity sector is of the standard Einstein-Hilbert form. The matter sector, however, becomes explicitly coupled to the inflaton sector, and we must also be careful to take into account the spacetime-dependent rescaling of units that results from the conformal rescaling of the metric.

B. Einstein's equations and the equations of motion

Having defined our actions, let us briefly review the gravitational equations of motion that they give rise to. In this section we will simply quote the main results. Additional details regarding these known results can be found in Appendix A.

Re-expressing (1) in the form

$$S = \int d^4x\sqrt{-g}\left\{\frac{f(\boldsymbol{\phi})}{2}R + \mathcal{L}^{(\phi)} + \mathcal{L}^{(m)}\right\}, \qquad \mathcal{L}^{(\phi)} = -\frac{1}{2}h_{ab}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b - V(\boldsymbol{\phi}), \tag{8}$$

where $\mathcal{L}^{(m)}$ contains the matter sector, and minimising (8) with respect to $g^{\mu\nu}$ we get

$$G_{\mu\nu} = \frac{1}{f}\left[T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu} + \nabla_\mu\nabla_\nu f - g_{\mu\nu}\Box f\right], \tag{9}$$

where

$$T^{(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(\phi)}\right)}{\delta g^{\mu\nu}} = h_{ab}\nabla_\mu\phi^a\nabla_\nu\phi^b - g_{\mu\nu}\left(\frac{1}{2}h_{ab}g^{\rho\sigma}\nabla_\rho\phi^a\nabla_\sigma\phi^b + V\right). \tag{10}$$

Similarly, varying the action with respect to the fields ϕ^a we get the equations of motion

$$h_{ab}\Box\phi^b + \Gamma_{bc|a}g^{\mu\nu}\nabla_\mu\phi^b\nabla_\nu\phi^c - V_a + f_aR = 0, \tag{11}$$

where $\Gamma_{ab|c} = h_{cd}\Gamma^d_{ab}$ and Γ^a_{bc} is the Christoffel connection associated with the field-space metric h_{ab}.

---

[2] Note that, for example, f_a denotes taking the derivative of f with respect to the a'th field.

Turning to the Einstein frame, we can similarly write the action (3) in the form

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{M_{\rm Pl}^2}{2}\tilde{R} + \tilde{\mathcal{L}}^{(\phi)} + \tilde{\mathcal{L}}^{(m)}\right\}, \qquad \tilde{\mathcal{L}}^{(\phi)} = -\frac{1}{2}S_{ab}\tilde{g}^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b - \tilde{V}, \tag{12}$$

and we will return shortly to the relation between $\tilde{\mathcal{L}}^{(m)}$ and $\mathcal{L}^{(m)}$. We then find the standard Einstein equations

$$\tilde{G}_{\mu\nu} = \frac{1}{M_{\rm Pl}^2}\left(\tilde{T}_{\mu\nu}^{(\phi)} + \tilde{T}_{\mu\nu}^{(m)}\right), \tag{13}$$

where

$$\tilde{T}_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta\left(\sqrt{-\tilde{g}}\tilde{\mathcal{L}}^{(\phi)}\right)}{\delta\tilde{g}^{\mu\nu}} = S_{ab}\tilde{\nabla}_\mu\phi^a\tilde{\nabla}_\nu\phi^b - \tilde{g}_{\mu\nu}\left(\frac{1}{2}S_{ab}\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\phi^a\tilde{\nabla}_\sigma\phi^b + \tilde{V}\right). \tag{14}$$

The equations of motion for the fields in the Einstein frame take the form

$$-S_{ab}\tilde{\Box}\phi^b - {}^{(S)}\Gamma_{bc|a}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi^b\tilde{\nabla}_\nu\phi^c + \tilde{V}_a + \frac{\Omega_a}{\Omega}\tilde{T}^{(m)} = 0, \tag{15}$$

where ${}^{(S)}\Gamma_{bc|a} = S_{ad}{}^{(S)}\Gamma^d_{bc}$ and ${}^{(S)}\Gamma^d_{bc}$ is the Christoffel connection associated with S_{ab}.

Regarding the matter energy-momentum tensors, one can show (see Appendix A for a review) that under certain conditions the following relations hold:

$$T_{\mu\nu}^{(m)} = \Omega^2\tilde{T}_{\mu\nu}^{(m)}, \qquad \nabla^\mu T_{\mu\nu}^{(m)} = 0 \qquad \text{and} \qquad \tilde{\nabla}^\mu\tilde{T}_{\mu\nu}^{(m)} = -\frac{\Omega_\nu}{\Omega}\tilde{T}^{(m)}. \tag{16}$$

For future reference we note that the energy-momentum tensors associated with χ_i and ψ_i are given, respectively, as

$$T_{\mu\nu}^{(\chi_i)} = \nabla_\mu\chi_i\nabla_\nu\chi_i - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\nabla_\rho\chi_i\nabla_\sigma\chi_i + U(\chi_i)\right), \tag{17}$$

$$T_{\mu\nu}^{(\psi_i)} = \frac{1}{2}\left(\overline{\psi}_i\gamma_{(\mu}(x)D_{\nu)}\psi_i - D_{(\mu}\overline{\psi}_i\gamma_{\nu)}(x)\psi_i\right), \tag{18}$$

where our symmetrisation with respect to the indices includes a factor of a half. In the Einstein frame we similarly have

$$\tilde{T}_{\mu\nu}^{(\tilde{\chi}_i)} = \mathcal{D}_\mu\tilde{\chi}_i\mathcal{D}_\nu\tilde{\chi}_i - \tilde{g}_{\mu\nu}\left(\frac{1}{2}\tilde{g}^{\rho\sigma}\mathcal{D}_\rho\tilde{\chi}_i\mathcal{D}_\sigma\tilde{\chi}_i + \frac{U(\tilde{\chi}_i)}{\Omega^4}\right), \tag{19}$$

$$\tilde{T}_{\mu\nu}^{(\tilde{\psi}_i)} = \frac{1}{2}\left(\overline{\tilde{\psi}}_i\tilde{\gamma}_{(\mu}(x)D_{\nu)}\tilde{\psi}_i - D_{(\mu}\overline{\tilde{\psi}}_i\tilde{\gamma}_{\nu)}(x)\tilde{\psi}_i\right). \tag{20}$$

III. REHEATING DYNAMICS

Having described the general setup for our class of models and the relation between the Jordan and Einstein frame formulations, in this section we turn to the process of reheating. We will begin by considering the background dynamics of the inflaton fields after the end of inflation, before then turning to the particle production process. We also consider the effect of the produced particles on the dynamics of the inflaton fields and how reheating ends. At every step we try to discuss how the interpretation of the reheating process differs in the Jordan and Einstein frames. For a review of reheating after inflation and the techniques employed in this section, see e.g. [17, 40, 41].

A. The oscillating phase

In describing the dynamics of the inflaton fields after the end of inflation we make the standard assumption that at background level our Universe is described by a Friedmann-Lemaître-Robertson-Walker (FLRW)

metric. Indeed, the homogeneity and isotropy of the Universe should be guaranteed thanks to the preceding epoch of inflation. If we would like to write both the Jordan and Einstein frame metrics in FLRW form, then the relation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ gives us

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})\delta_{ij}d\tilde{x}^i d\tilde{x}^j = \Omega^2 ds^2 = \Omega^2\left(-dt^2 + a^2(t)\delta_{ij}dx^i dx^j\right), \tag{21}$$

where we recall that $\Omega^2 = f(\phi)/M_{\rm Pl}^2$. From the above equation we then find the following relations:

$$\tilde{a} = \Omega a, \quad d\tilde{t} = \Omega dt, \quad d\tilde{x}^i = dx^i \quad \text{and} \quad \tilde{H} = \frac{1}{\Omega}\Big(H + \frac{\dot{\Omega}}{\Omega}\Big). \tag{22}$$

If we consider the epoch before reheating, when only the inflaton sector is present, then the Friedmann equation and equations of motion for the scalar fields in the Jordan frame are given, respectively, as

$$3H^2 = \frac{1}{f}\left[\frac{1}{2}h_{ab}\dot{\phi}^a\dot{\phi}^b + V - 3H\dot{f}\right] \qquad \text{and} \qquad \frac{D\dot{\phi}^a}{dt} + 3H\dot{\phi}^a + h^{ab}\left(V_b - f_b R\right) = 0, \tag{23}$$

where $D\dot{\phi}^a/dt = \ddot{\phi}^a + \Gamma^a_{bc}\dot{\phi}^b\dot{\phi}^c$ and a dot denotes a derivative with respect to the Jordan frame cosmic time t. Similarly, in the Einstein frame we have

$$3\tilde{H}^2 = \frac{1}{M_{\rm Pl}^2}\left[\frac{1}{2}S_{ab}\frac{d\phi^a}{d\tilde{t}}\frac{d\phi^b}{d\tilde{t}} + \tilde{V}\right] \qquad \text{and} \qquad \frac{\tilde{D}(d\phi^a/d\tilde{t})}{d\tilde{t}} + 3\tilde{H}\frac{d\phi^a}{d\tilde{t}} + S^{ab}\tilde{V}_b = 0, \tag{24}$$

where $\tilde{D}(d\phi^a/d\tilde{t})/d\tilde{t} = d^2\phi^a/d\tilde{t}^2 + {}^{(S)}\Gamma^a_{bc}(d\phi^b/d\tilde{t})(d\phi^c/d\tilde{t})$, $\tilde{V} = M_{\rm Pl}^4 V/f^2$ and $\tilde{t}$ denotes the cosmic time in the Einstein frame.

Given that the non-minimal coupling between the inflaton fields and gravity is removed in transforming to the Einstein frame, it is much more convenient to solve for the inflaton dynamics in this frame. As such, let us proceed by first solving for the dynamics in the Einstein frame. Do note, however, that it should be possible to solve directly in the Jordan frame, see e.g. [32].

Given that in the Einstein frame the dynamics are determined by the Einstein frame potential $\tilde{V}$ and the field-space curvature associated with S_{ab}, we make the assumption that at the end of inflation all of the inflaton fields begin to oscillate about the minimum of $\tilde{V}$ at $\phi^a = \phi^a_{\rm vev}$, and can be decomposed as $\phi^a = \phi^a_{\rm vev} + \sigma^a$. Requiring the absence of a cosmological constant dictates that $\tilde{V}_{\rm vev} = 0$. Combining this with the fact that $\tilde{V}_a|_{\rm vev} = 0$, on expanding the inflaton part of the Einstein-frame action to second order in σ^a we get

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{M_{\rm Pl}^2\tilde{R}}{2} - \frac{1}{2}S_{ab}|_{\rm vev}\tilde{g}^{\mu\nu}\partial_\mu\sigma^a\partial_\nu\sigma^b - \frac{1}{2}\tilde{V}_{ab}|_{\rm vev}\sigma^a\sigma^b\right\}, \tag{25}$$

where we have made the assumption that the potential can be well approximated as being quadratic about its minimum. In order to deal with the non-diagonal nature of this action, we now introduce the mass eigenstates of the Einstein-frame potential. Namely, we take $\sigma^a = e^a_A\alpha^A$, where

$$\tilde{V}^a{}_b|_{\rm vev}e^b_A = m^2_{\hat{A}}e^a_{\hat{A}}, \tag{26}$$

with $\tilde{V}^a{}_b|_{\rm vev} = S^{ac}|_{\rm vev}\tilde{V}_{cb}|_{\rm vev}$ and $S_{ab}|_{\rm vev}e^a_A e^b_B = \delta_{AB}$. In the above expression the hat on the index A suppresses summation. The Einstein frame action then takes the form

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{M_{\rm Pl}^2\tilde{R}}{2} + \frac{1}{2}\sum_A\left[\left(\frac{d\alpha^A}{d\tilde{t}}\right)^2 - m_A^2(\alpha^A)^2\right]\right\}, \tag{27}$$

so that the equations of motion for α^A are simply given as

$$\frac{d^2}{d\tilde{t}^2}\left(\tilde{a}^{3/2}\alpha^A\right) + \left[m^2_{\hat{A}} - \left(\frac{9}{4}\tilde{H}^2 + \frac{3}{2}\frac{d\tilde{H}}{d\tilde{t}}\right)\right]\left(\tilde{a}^{3/2}\alpha^A\right) = 0. \tag{28}$$

Note that in deriving the above results we have taken σ^a and hence α^A to be background quantities that only depend on time, i.e. the decomposition $\phi^a = \phi^a_{\rm vev} + \sigma^a$ is not a decomposition of ϕ^a into its classical

background part and quantum perturbation, but is simply an expansion of the classical part of ϕ^a about $\phi^a_{\rm vev}$. Also note that no terms involving the field-space curvature appear in (28). This is because such terms would be second order in σ^a, taking the form ${}^{(S)}\Gamma^a_{bc}|_{\rm vev}(d\sigma^b/d\tilde{t})(d\sigma^c/d\tilde{t})$. As such, we see that models with very large field-space curvature, such as that considered in Fig. 5 of [10], lie beyond the scope of our perturbative approach.

Making the assumption $m_A^2 \gg \tilde{H}^2$, $d\tilde{H}/d\tilde{t}$, i.e. that the timescale of the field oscillations is much shorter than that of the background evolution of the universe, we find the solutions

$$\alpha^A \simeq \frac{\alpha_0^A}{\tilde{a}^{3/2}}\cos[m_{\hat{A}}\tilde{t} + d_{\hat{A}}], \tag{29}$$

where d_A are constant phases. The Einstein frame Friedmann equation then gives us

$$\tilde{H}^2 = \frac{1}{6M_{\rm Pl}^2}\sum_A\left[\left(\frac{d\alpha^A}{d\tilde{t}}\right)^2 + m_A^2(\alpha^A)^2\right] = \sum_A \frac{(\alpha_0^A)^2 m_A^2}{6M_{\rm Pl}^2\tilde{a}^3}\left(1 + \frac{3\tilde{H}}{2m_A}\sin(2(m_A\tilde{t}+d_A)) + \mathcal{O}(\tilde{H}^2/m_A^2)\right). \tag{30}$$

As such, we see that to leading order in $\tilde{H}/m_A$ the evolution of the Einstein frame Hubble rate coincides with that of a matter-dominated universe. On calculating $d\tilde{H}/d\tilde{t}$ one finds

$$\frac{d\tilde{H}}{d\tilde{t}} \simeq -\frac{3}{2}\tilde{H}^2\left(1 - \frac{1}{\tilde{H}^2}\sum_A\frac{(\alpha_0^A)^2 m_A^2}{6M_{\rm Pl}^2\tilde{a}^3}\cos(2(m_A\tilde{t}+d_A))\right). \tag{31}$$

The second term in the brackets represents an $\mathcal{O}(1)$ deviation from the case of matter-domination, which can be seen by noting from (30) that $(\alpha_0^A)^2 m_A^2/(6M_{\rm Pl}^2\tilde{a}^3) \sim \mathcal{O}(\tilde{H}^2)$. However, the important result as far as we are concerned is that $d\tilde{H}/d\tilde{t} \sim \tilde{H}^2 \ll m_A^2$.

We are now interested in using these results to determine the background evolution in the Jordan frame. As is evident from (22), in order to do this we need expressions for f and its derivatives, and these can be obtained by expanding f about $f_{\rm vev}$. In doing so, we make the assumption that by the end of reheating, when all fields have decayed, $f_{\rm vev} = M_{\rm Pl}^2$. We therefore have

$$f = M_{\rm Pl}^2\left(1 + \frac{f_a\sigma^a}{M_{\rm Pl}^2} + \frac{1}{2}\frac{f_{ab}\sigma^a\sigma^b}{M_{\rm Pl}^2} + ...\right) = M_{\rm Pl}^2\left(1 + \frac{f_A\alpha^A}{M_{\rm Pl}^2} + \frac{1}{2}\frac{f_{AB}\alpha^A\alpha^B}{M_{\rm Pl}^2} + ...\right), \tag{32}$$

where $f_A = f_a e^a_A$. Inserting this expansion into the last relation in (22) and evaluating to leading order in α^A and $\tilde{H}/m_A$ we get

$$H \simeq \tilde{H}\left(1 + \frac{1}{\tilde{H}}\sum_A \frac{f_A}{2M_{\rm Pl}^2}\frac{\alpha_0^A m_A}{\tilde{a}^{3/2}}\sin\left(m_A\tilde{t} + d_A\right)\right). \tag{33}$$

If we assume that $f_A/M_{\rm Pl} \sim \mathcal{O}(1)$, and recall from (30) that $\alpha_0^A m_A/(M_{\rm Pl}\tilde{a}^{3/2}) \sim \mathcal{O}(\tilde{H})$, we see that the evolution of the Hubble rate in the Jordan frame has an oscillatory component that is not suppressed. Note that to leading order in $\tilde{H}/m_A$ the cosmic times as defined in the Jordan and Einstein frames are interchangeable. With this, we see that $\dot{H}$ picks up a term that is $\mathcal{O}(m_A\tilde{H})$ (assuming $f_A/M_{\rm Pl} \sim \mathcal{O}(1)$). This is to be compared with the case in the Einstein frame, where $d\tilde{H}/d\tilde{t} \sim \mathcal{O}(\tilde{H}^2)$.

B. Perturbative QFT approach to reheating

Having discussed the background dynamics of the oscillating inflaton fields at the end of inflation, there are essentially two ways in which we can now consider reheating into ordinary matter. The first follows the standard perturbative QFT approach, and appears natural in the Einstein frame. The second method involves calculating Bogoliubov coefficients in an approach based on QFT in a time-varying classical background. This second method appears natural in whichever frame we begin, but the interpretation in each frame is somewhat different. In the case of perturbative reheating both methods are equally valid, and the result is independent of the method used.

Decay rates

In transforming to the Einstein frame, one consequence of the conformal transformation is that we explicitly see the appearance of interaction terms between the inflaton sector and ordinary matter. These are apparent in the factors of Ω that appear in $\mathcal{D}_\mu$, U/Ω^4 and m_{ψ_i}/Ω in (5) and (6). Using the expansion of f given in (32), and taking

$$U(\chi) = \frac{m_\chi^2 \chi^2}{2}, \tag{34}$$

we find that the Einstein frame action contains the tri-linear interaction terms

$$\mathcal{L}^{\chi}_{int} = \frac{f_A \alpha^A}{4M_{\rm Pl}^2}\left(2m_\chi^2 + m_{\hat{A}}^2\right)\tilde{\chi}^2 \qquad \text{and} \qquad \mathcal{L}^{\psi}_{int} = \frac{f_A\alpha^A}{2M_{\rm Pl}^2} m_\psi \bar{\tilde{\psi}}\tilde{\psi}, \tag{35}$$

where we have integrated by parts and used the equations of motion for α^A in deriving the first of these.[3] Here we neglect to consider four-point interaction terms, as we know that such terms cannot allow for complete reheating [40]. (See, however, Sec. IV of [29].)

Another key feature of the Einstein frame is that the scale factor is evolving slowly, i.e. $\tilde{H}^2$, $d\tilde{H}/d\tilde{t} \ll m_A^2$, which allows us to neglect the expansion of the Universe. As such, we can use flat-space QFT calculations to determine the transition amplitudes for $\alpha^A \to \tilde{\chi}\tilde{\chi}$ and $\alpha^A \to \bar{\tilde{\psi}}\tilde{\psi}$ that result from the interaction terms in (35). These amplitudes can in turn be used to calculate the decay rates per unit time and volume of the oscillating fields [27]:

$$\tilde{\Gamma}_{\alpha^A \to \chi\chi} = \frac{\tilde{g}_{\chi A}^2}{8\pi m_{\hat{A}}}\left(1 - \frac{4m_\chi^2}{m_{\hat{A}}^2}\right)^{1/2} \qquad \text{and} \qquad \tilde{\Gamma}_{\alpha^A \to \overline{\psi}\psi} = \frac{\tilde{g}_{\psi A}^2 m_{\hat{A}}}{8\pi}\left(1 - \frac{4m_\psi^2}{m_{\hat{A}}^2}\right)^{3/2}, \tag{36}$$

where

$$\tilde{g}_{\chi A} = \frac{f_a e^a_A (m_{\hat{A}}^2 + 2m_\chi^2)}{4M_{\rm Pl}^2} \qquad \text{and} \qquad \tilde{g}_{\psi A} = \frac{f_a e^a_A m_\psi}{2M_{\rm Pl}^2}. \tag{37}$$

In this approach we interpret the oscillating inflaton fields as a condensate of zero-momentum particles that can decay into two scalars or a fermion-anti-fermion pair. Our reason for suggesting that this approach seems "natural" in the Einstein frame is that it is in this frame that the necessary interaction terms are explicit and that the background evolution of the scale factor can be neglected.

Dynamics including decay products

Once the rate of decay becomes significant, namely $\tilde{\Gamma}_A \sim \tilde{H}$ (see (39) for the definition of $\tilde{\Gamma}_A$), we must take into account the effect that the decay products have on the oscillating inflaton dynamics. Remaining in the Einstein frame, we see from (15) that the dynamics of the inflaton fields is sourced by the trace of the matter energy-momentum tensor. In Sec. III A we ignored this term, assuming that inflaton decay was initially negligible, but now we must properly include it. At the level of the action the effect of matter fields on the dynamics of α^A is evident in the explicit interaction terms, such as those given in (35). As such, in the context of the perturbative QFT approach it is necessary to calculate 1-loop corrections to the propagator of α^A. Invoking the optical theorem, one finds that the effective equations of motion for α^A take the form [40]

$$\frac{d^2}{d\tilde{t}^2}\left(\tilde{a}^{3/2}\alpha^A\right) + \left[m_{\hat{A}}^2 + i m_{\hat{A}}\tilde{\Gamma}_{\hat{A}} - \left(\frac{9}{4}\tilde{H}^2 + \frac{3}{2}\frac{d\tilde{H}}{d\tilde{t}}\right)\right]\left(\tilde{a}^{3/2}\alpha^A\right) = 0, \tag{38}$$

where

$$\tilde{\Gamma}_A = \sum_i \tilde{\Gamma}_{\alpha^A \to \chi_i\chi_i} + \sum_j \tilde{\Gamma}_{\alpha^A \to \overline{\psi}_j\psi_j}. \tag{39}$$

---

[3] Note that as we have used the background equations of motion for α^A, the effective interaction term is only valid in making tree-level calculations. If we wish to go beyond tree-level calculations, then we would have to use the derivative interaction term directly.

On inserting the zeroth-order solution for $\tilde{H}$, namely $\tilde{H} = 2/3\tilde{t}$, the last term in the square brackets vanishes. If we also assume $m_A \gg \tilde{\Gamma}_A$, then the solutions to the above equations take the form

$$\alpha^A = \frac{\alpha_0^A}{\tilde{a}^{3/2}} \exp\left[-\frac{1}{2}\tilde{\Gamma}_{\hat{A}}\tilde{t}\right] \cos\left[m_{\hat{A}}\tilde{t} + d_{\hat{A}}\right]. \tag{40}$$

Comparing with (29), we see that there is an additional exponential decay of the amplitude of the oscillations.

Phenomenologically, the effect of inflaton decay is often modeled by including an additional frictional term in the equations of motion for α^A as follows [40]:

$$\frac{d^2\alpha^A}{d\tilde{t}^2} + \left(3\tilde{H} + \tilde{\Gamma}_{\hat{A}}\right)\frac{d\alpha^A}{d\tilde{t}} + m_{\hat{A}}\alpha^A = 0. \tag{41}$$

Indeed, under the assumptions $m_A \gg \tilde{H}, \Gamma_A$, one can see that (40) does satisfy this equation. The advantage of using this phenomenological equation is that it can be recast in a form that is intuitive. On multiplying through by $d\alpha^A/d\tilde{t}$ and averaging over many cycles, it can be re-written as

$$\frac{d\tilde{\rho}_A}{d\tilde{t}} + 3\tilde{H}\tilde{\rho}_A + \tilde{\Gamma}_{\hat{A}}\tilde{\rho}_A = 0, \tag{42}$$

where we have once again assumed $m_A \gg \tilde{H}, \Gamma_A$ and

$$\tilde{\rho}_A = \frac{1}{2}\left(\frac{d\alpha^A}{d\tilde{t}}\right)^2 + \frac{1}{2}m_{\hat{A}}^2(\alpha^A)^2. \tag{43}$$

Summing over all A we have

$$\frac{d\tilde{\rho}_\alpha}{d\tilde{t}} + 3\tilde{H}\tilde{\rho}_\alpha + \tilde{\Gamma}_{\hat{\alpha}}\tilde{\rho}_\alpha = 0, \tag{44}$$

where

$$\tilde{\rho}_\alpha = \sum_A \tilde{\rho}_A \qquad \text{and} \qquad \tilde{\Gamma}_\alpha = \sum_A \frac{\tilde{\rho}_A}{\tilde{\rho}_\alpha}\tilde{\Gamma}_A. \tag{45}$$

We thus see that the energy density of the oscillating fields decays as a result of the Hubble expansion and the decay into matter particles, which is intuitively what we expect. It is important to note, however, that this phenomenological approach relies on the nature of the interaction terms considered and the fact that the inflaton fields are oscillating in a quadratic potential. As such, the situation will be different in the more general case [42].

Assuming that the decay products quickly thermalise and can be modeled as a relativistic fluid, we also have the equations

$$\frac{d\tilde{\rho}_{\chi_i}}{d\tilde{t}} + 4\tilde{H}\tilde{\rho}_{\chi_i} - \sum_A \tilde{\Gamma}_{\alpha^A \to \chi_i\chi_i}\tilde{\rho}_A = 0, \tag{46}$$

$$\frac{d\tilde{\rho}_{\psi_i}}{d\tilde{t}} + 4\tilde{H}\tilde{\rho}_{\psi_i} - \sum_A \tilde{\Gamma}_{\alpha^A \to \overline{\psi}_i\psi_i}\tilde{\rho}_A = 0. \tag{47}$$

Combining with (42), one can see that the total energy density is thus covariantly conserved, in agreement with (13).

C. Bogoliubov approach to reheating

In the flat-space perturbative QFT approach discussed in the previous subsection, we considered the oscillating inflaton fields as a collection of massive zero-momentum particles decaying into matter fields. One of the limitations of this approach is that it can only be applied in the perturbative regime, where interaction terms are small. An alternative approach to calculating the decay rates is based on QFT in a time-varying classical background. Within this framework particle production is a collective phenomenon, and the interaction terms do not necessarily have to be small. In the case that they are small, the results of the previous subsection are recovered, but in the case that the interaction terms are large it is possible to obtain resonant particle production, or preheating – see e.g. [40]. In this paper we will focus on the perturbative regime and confirm agreement with the perturbative QFT results given in the previous subsection. Much of our discussion, however, will also be relevant in the preheating regime.

In calculating particle production in a time-varying classical background one is interested in solving for the mode functions of the matter field under consideration, [43–46], and it is important that these are the mode functions associated with canonically normalised fields. As a consequence, we find that the calculation becomes independent of the frame in which one starts, leaving only a difference in interpretation. To demonstrate this let us consider the case of the bosonic field χ.

Specialising to the case of an FLRW metric, the Jordan frame action for χ becomes

$$S_\chi = \int dt d^3 x a^3 \frac{1}{2}\left[\dot{\chi}^2 - \frac{1}{a^2}(\nabla\chi)^2 - m_\chi^2\chi^2\right], \tag{48}$$

where we have assumed $U(\chi) = m_\chi^2\chi^2/2$. In order to bring this into canonical form we use conformal time – defined as $a d\eta = dt$ – and also introduce the re-scaled field $u = a\chi$, giving

$$S_u = \int d\eta d^3 x \frac{1}{2}\left[u'^2 - (\nabla u)^2 - \left(a^2 m_\chi^2 - \frac{a''}{a}\right)u^2\right], \tag{49}$$

where a prime denotes differentiation with respect to conformal time. Working in Fourier space this gives rise to the equations of motion for the mode functions as

$$u_k'' + w_k^2 u_k = 0 \qquad \text{with} \qquad w_k^2 = k^2 + a^2 m_\chi^2 - \frac{a''}{a}, \tag{50}$$

where $k = |\vec{k}|$. As already discussed in Sec. III A, the scale factor in the Jordan frame has a rapidly oscillating component, and it is the resultant rapid time-variation of the effective mass of u_k that gives rise to particle production.

If we now go to the Einstein frame, the action for $\tilde{\chi}$ takes the form

$$S_{\tilde{\chi}} = \int d\tilde{t} d^3 x \tilde{a}^3 \frac{1}{2}\left[\left(\frac{d\tilde{\chi}}{d\tilde{t}}\right)^2 - \frac{1}{\tilde{a}^2}(\nabla\tilde{\chi})^2 - m_\chi^2\tilde{\chi}^2 + \frac{f_A\alpha^A}{2M_{\rm Pl}^2}\left(2m_\chi^2 + m_{\hat{A}}^2\right)\tilde{\chi}^2\right], \tag{51}$$

where we have expanded f to linear order in α^A, integrated by parts and used the equations of motion for α^A in order to get the interaction terms. In order to bring this into canonical form we once again use conformal time (note that conformal time is frame independent) and define the field $\tilde{u} = \tilde{a}\tilde{\chi}$. However, given that $\tilde{\chi} = \chi/\Omega$, we see that

$$\tilde{u} = \tilde{a}\tilde{\chi} = \frac{\tilde{a}}{\Omega}\chi = a\chi = u. \tag{52}$$

As such, whichever frame we start in, the analysis becomes identical once we transform to the canonically normalised variables. Indeed, on using (22) one finds

$$\frac{a''}{a} = a^2\left(\dot{H} + 2H^2\right) = \frac{\tilde{a}^2 M_{\rm Pl}^2}{f}\left[\frac{f}{M_{\rm Pl}^2}\left(\frac{d\tilde{H}}{d\tilde{t}} + \tilde{H}^2\right) - \frac{1}{2M_{\rm Pl}^2}\left(\frac{d^2 f}{d\tilde{t}^2} + 3\tilde{H}\frac{df}{d\tilde{t}}\right) + \frac{3}{4fM_{\rm Pl}^2}\left(\frac{df}{d\tilde{t}}\right)^2\right]. \tag{53}$$

Expanding f to first order in α^A and using the equations of motion for α^A this reduces to

$$\frac{a''}{a} = \frac{\tilde{a}''}{\tilde{a}} + \frac{f_A\alpha^A}{2M_{\rm Pl}^2}\tilde{a}^2 m_{\hat{A}}^2 + \mathcal{O}((\alpha^A)^2), \tag{54}$$

so that on substituting into (50) we have

$$w_k^2 \simeq k^2 + \tilde{a}^2 m_\chi^2 - \frac{\tilde{a}''}{\tilde{a}} - \frac{f_A\alpha^A}{M_{\rm Pl}^2}\tilde{a}^2 m_\chi^2 - \frac{1}{2}\frac{f_A\alpha^A}{M_{\rm Pl}^2}\tilde{a}^2 m_{\hat{A}}^2, \tag{55}$$

and this is in agreement with the frequency we would obtain from (51) on defining $\tilde{u} = \tilde{a}\tilde{\chi}$.

The only difference between the two frames, therefore, is the interpretation. In the Jordan frame the effective mass of the scalar field is oscillating as a result of the oscillating scale factor, which is why we refer

to the process as gravitational reheating. In the Einstein frame, however, the scale factor is slowly varying and the oscillatory nature of the effective mass of the scalar simply results from the explicit interaction terms in the action.

Whilst above we have considered the case of a scalar field, one can easily see that the same applies for fermions. On using conformal time, in the Jordan frame the canonically normalised field is $\Psi = a^{3/2}\psi$. In the Einstein frame, on the other hand, we have

$$\tilde{\Psi} = \tilde{a}^{3/2}\tilde{\psi} = \left(\frac{\tilde{a}}{\Omega}\right)^{3/2}\psi = a^{3/2}\psi = \Psi. \tag{56}$$

The production of bosons

We now turn to calculating the particle production rate for bosons, and we follow very closely the analyses given in [40, 45–47]. Let us start by expanding the quantum operator $u = a\chi$ in the standard way as

$$u = \int \frac{d^3k}{(2\pi)^{3/2}}\left[u_k a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + u_k^* a_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}}\right], \tag{57}$$

where u_k satisfy (50) and $a_{\vec{k}}^\dagger$ and $a_{\vec{k}}$ are the creation and annihilation operators satisfying the standard commutation relations.

The Hamiltonian associated with the action (49) can then be expanded as

$$\mathrm{H}_u = \frac{1}{2}\int d^3k\left[2E_k\left(2a_{\vec{k}}^\dagger a_{\vec{k}} + \delta^{(3)}(0)\right) + F_k a_{\vec{k}} a_{-\vec{k}} + F_k^* a_{\vec{k}}^\dagger a_{-\vec{k}}^\dagger\right], \tag{58}$$

where

$$2E_k = |u_k'|^2 + w_k^2|u_k|^2 \qquad \text{and} \qquad F_k = (u_k')^2 + w_k^2 u_k^2. \tag{59}$$

It is possible to construct mode functions that satisfy $F_k = 0$ – thus diagonalising the Hamiltonian – as

$$u_k(\eta) = \frac{\alpha_k(\eta)}{\sqrt{2w_k(\eta)}}\exp\left[-i\int_{-\infty}^{\eta} d\eta' w_k(\eta')\right] + \frac{\beta_k(\eta)}{\sqrt{2w_k(\eta)}}\exp\left[i\int_{-\infty}^{\eta} d\eta' w_k(\eta')\right], \tag{60}$$

where $\alpha_k(\eta)$ and $\beta_k(\eta)$ must satisfy the equations

$$\alpha_k'(\eta) = \frac{w_k'}{2w_k}\exp\left[2i\int_{-\infty}^{\eta} d\eta' w_k(\eta')\right]\beta_k(\eta), \tag{61}$$

$$\beta_k'(\eta) = \frac{w_k'}{2w_k}\exp\left[-2i\int_{-\infty}^{\eta} d\eta' w_k(\eta')\right]\alpha_k(\eta). \tag{62}$$

We also require that $|\alpha_k|^2 - |\beta_k|^2 = 1$ in order that the canonical commutation relations for u are satisfied. One can confirm that these mode functions also satisfy the equations of motion, and that $E_k = w_k(1/2 + |\beta_k(\eta)|^2)$. If in the asymptotic past $w_k(\eta_0)$ is approximately constant, then the mode functions defined above are a linear combination of the positive- and negative-frequency mode functions associated with the Bunch-Davies vacuum. If $\beta_k(\eta_0) = 0$ at this time, then the mode functions do indeed coincide with those of the Bunch-Davies vacuum, where E_k is minimised. As time evolves, however, the evolution of $w_k(\eta)$ causes $\beta_k(\eta)$ to evolve away from zero, meaning that E_k is no longer minimised. Given the diagonal nature of the Hamiltonian, this effect can be interpreted as particle production.

Let us now assume that at the initial time η_0 at which inflaton oscillations commenced there are no χ particles present, i.e. $\beta_k(\eta_0) = 0$ and $\alpha_k(\eta_0) = 1$. We then assume that at times shortly after η_0 we are in the perturbative regime where $\beta_k(\eta) \ll 1$ and $\alpha_k(\eta) - 1 \ll 1$. This ensures that we are in the same perturbative regime for which the QFT calculations of Sec. III B are applicable, where Bose condensate effects are neglected. Solving the above relations iteratively, we find a solution for $\beta_k(\eta)$ as

$$\beta_k(\eta) \simeq \int_{\eta_0}^{\eta} d\eta' \frac{w_k'}{2w_k}\exp\left[-2i\int_{-\infty}^{\eta'} d\eta'' w_k(\eta'')\right]. \tag{63}$$

Using (55), we find that w_k' and w_k^2 to leading order in α^A and $\tilde{H}/m_A$ are given as

$$w_k' \simeq -\frac{\tilde{a}^2}{2w_k}\frac{f_A\alpha^{A\prime}}{2M_{\rm Pl}^2}\left(m_{\hat{A}}^2+2m_\chi^2\right)+\frac{\tilde{a}^3\tilde{H}m_\chi^2}{w_k}, \tag{64}$$

$$w_k^2 \simeq k^2+\tilde{a}^2m_\chi^2. \tag{65}$$

Taking expressions to leading order in α^A ensures that we are only considering the tri-linear interaction terms and perturbative regime appropriate for comparison with the perturbative QFT calculations of Sec. III B. In making order-of-magnitude estimates, we note that from (30) one can deduce the order-of-magnitude relations $\alpha'/M_{\rm Pl}\sim\tilde{a}\tilde{H}$ and $m_{\hat{A}}\alpha^A/M_{\rm Pl}\sim\tilde{H}$. We have also assumed that $f_A/M_{\rm Pl}\sim\mathcal{O}(1)$ and $k\sim\mathcal{O}(\tilde{a}m_A)$. The first of these assumptions is compatible with expansion (32) so long as we have $\alpha^A/M_{\rm Pl}\ll 1$, and the second assumption comes from our expectation that modes with $k\sim\mathcal{O}(\tilde{a}m_A)$ will be produced. On substituting (64), (65) and (29) into (63) we notice that in general the integrand is highly oscillatory in η. As the second term in (64) is non-oscillatory, we find that its contribution to $\beta_k(\eta)$ averages to zero. The first term in (64), however, is oscillatory, which allows for the possibility of stationary points in the total phase of the integrand. We thus have

$$\beta_k(\eta)=\sum_A\frac{m_Af_A\alpha_0^A(2m_\chi^2+m_A^2)}{16M_{\rm Pl}^2 i}\int_{\eta_0}^{\eta}d\eta'\frac{\tilde{a}^{3/2}(\eta')}{k^2+\tilde{a}^2(\eta')m_\chi^2}\left\{\exp[im_A\psi_{k,1}^A(\eta')]-\exp\left[im_A\psi_{k,2}^A(\eta')\right]\right\}, \tag{66}$$

where

$$\psi_{k,1}^A(\eta')=\int_{-\infty}^{\eta'}d\eta''\left(\tilde{a}-\frac{2w_k}{m_A}\right)+\frac{d_A}{m_{\hat{A}}}, \tag{67}$$

$$\psi_{k,2}^A(\eta')=-\int_{-\infty}^{\eta'}d\eta''\left(\tilde{a}+\frac{2w_k}{m_A}\right)-\frac{d_A}{m_{\hat{A}}}. \tag{68}$$

The phase $\psi_{k,2}^A(\eta)$ does not have a stationary point for physical values of $\tilde{a}$, so only the term involving $\psi_{k,1}^A(\eta)$ contributes to $\beta_k(\eta)$. Using the stationary phase approximation we find

$$\beta_k(\eta)=\sum_A\beta_k^A\exp\left[im_A\psi_{k,1}^A(\eta_k^A)+is_k^A\pi/4\right]\Theta(\eta-\eta_k^A)\Theta(\eta_k^A-\eta_0), \tag{69}$$

$$\text{where}\qquad \beta_k^A=\frac{m_{\hat{A}}f_{\hat{A}}\alpha_0^A(2m_\chi^2+m_{\hat{A}}^2)}{16M_{\rm Pl}^2 i}\frac{4}{m_{\hat{A}}^2}\sqrt{\frac{2\pi}{\tilde{a}(\eta_k^{\hat{A}})m_{\hat{A}}|\psi_{k,1}^{\hat{A}\prime\prime}(\eta_k^{\hat{A}})|}}, \tag{70}$$

η_k^A is the time at which $d\psi_{k,1}^A/d\eta=0$ for some given k and s_k^A is the sign of $\psi_{k,1}^{A\prime\prime}(\eta_k^A)$. The phase is stationary when $2w_k=\tilde{a}m_A$, which to leading order gives

$$\frac{k^2}{\tilde{a}^2(\eta_k^A)}\simeq\frac{m_A^2}{4}\left(1-\frac{4m_\chi^2}{m_{\hat{A}}^2}\right). \tag{71}$$

Given that $k/\tilde{a}(\eta_k^A)$ coincides with the momentum of the produced particle, this result coincides with our expectation from kinematic considerations. The second derivative of the phase is given as

$$\psi_{k,1}^{A\prime\prime}(\eta_k^{\hat{A}})\simeq\tilde{a}^2(\eta_k^{\hat{A}})\tilde{H}(\eta_k^{\hat{A}})\left(1-\frac{4m_\chi^2}{m_A^2}\right)+\frac{2\tilde{a}(\eta_k^{\hat{A}})}{m_{\hat{A}}^2}\sum_B\frac{f_B\alpha^{B\prime}(\eta_k^{\hat{A}})}{2M_{\rm Pl}^2}\left(m_B^2+2m_\chi^2\right). \tag{72}$$

The two step functions in the above expression for $\beta_k(\eta)$ simply reflect the fact that a certain mode will only have been excited if $\eta>\eta_k^A>\eta_0$. Note that η_k^A is different for different A.

In looking to determine the production rate of χ particles let us start by considering the continuity equation for the Einstein frame energy-momentum tensor associated with $\tilde{\chi}$. Under the assumption of a FLRW background, we know that the expectation value of the energy-momentum tensor can be expressed as $\langle 0|\tilde{T}^{(\tilde{\chi})\mu}{}_\nu|0\rangle=\mathrm{diag}(-\tilde{\rho}_\chi,\ \tilde{p}_\chi,\ \tilde{p}_\chi,\ \tilde{p}_\chi)$, where $|0\rangle$ is the vacuum state as defined with respect to $a_{\vec{k}}$. The continuity equation (16) can then be written as

$$\frac{1}{\tilde{a}^4}\frac{d}{d\tilde{t}}(\tilde{a}^4\tilde{\rho}_\chi)+\tilde{H}(-\tilde{\rho}_\chi+3\tilde{p}_\chi)=\frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\chi+3\tilde{p}_\chi). \tag{73}$$

In the absence of particle production, i.e. in the absence of the interaction terms given in (35), the right-hand side of this equation would be vanishing. As such, in order to determine the particle production rate, we wish to evaluate the right-hand side of (73).

Using (19) to determine $\tilde{T}^{(\tilde{\chi})\mu}{}_{\nu}$, re-expressing the result in terms of the canonically normalised field u and taking the vacuum expectation value, we find

$$\begin{aligned}\tilde{\rho}_\chi &= \frac{1}{(2\pi)^3\tilde{a}^4}\int d^3k\left[w_k\left(\frac{1}{2}+|\beta_k|^2\right)-\mathcal{H}\Im\left(\alpha_k\beta_k^* e^{-2i\int w_k d\eta'}\right)\right.\\&\qquad\left.+\frac{1}{2w_k}(\mathcal{H}'+2\mathcal{H}^2)\left(\frac{1}{2}+|\beta_k|^2+\Re\left(\alpha_k\beta_k^* e^{-2i\int w_k d\eta'}\right)\right)\right],\end{aligned}\tag{74}$$

$$\begin{aligned}-\tilde{\rho}_\chi+3\tilde{p}_\chi &= -\frac{1}{(2\pi)^3\tilde{a}^4}\int d^3k\left[2w_k\Re\left(\alpha_k\beta_k^* e^{-2i\int w_k d\eta'}\right)+2\mathcal{H}\Im\left(\alpha_k\beta_k^* e^{-2i\int w_k d\eta'}\right)\right.\\&\qquad\left.+\frac{1}{w_k}\left(\mathcal{H}'+m_\chi^2 a^2\right)\left(\frac{1}{2}+|\beta_k|^2+\Re\left(\alpha_k\beta_k^* e^{-2i\int w_k d\eta'}\right)\right)\right],\end{aligned}\tag{75}$$

where $\mathcal{H}=a'/a$, $\Re(X)$ denotes the real part of X and $\Im(X)$ similarly the imaginary part. Assuming that $\Im\left(\alpha_k\beta_k^* e^{-2i\int w_k d\eta'}\right)$ is of the same order of magnitude as $\Re\left(\alpha_k\beta_k^* e^{-2i\int w_k d\eta'}\right)$, keeping terms only linear in β_k and neglecting the vacuum density contribution, we find that to lowest order in $\tilde{\mathcal{H}}/w_k$ the right-hand side of (73) is given as

$$\frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\chi+3\tilde{p}_\chi)\simeq-\frac{1}{(2\pi)^3\tilde{a}^4}\frac{1}{2f}\frac{df}{d\tilde{t}}\int d^3k\frac{1}{w_k}\left[2w_k^2+m_\chi^2\tilde{a}^2\right]\Re\left(\alpha_k\beta_k^* e^{-2i\int w_k d\eta'}\right).\tag{76}$$

In general this quantity is highly oscillatory, and we are therefore interested in finding its average over several oscillations, i.e. over a time-scale $T\sim\mathcal{O}(1/(\tilde{a}m_A))$, where we have chosen to work in conformal time. Taking $\alpha_k\simeq 1$ and substituting the results (29) and (69) we have

$$\begin{aligned}\left\langle\frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\chi+3\tilde{p}_\chi)\right\rangle &= \frac{1}{2T}\int_{\eta-T}^{\eta+T}d\eta'\sum_A\frac{4\pi}{(2\pi)^3\tilde{a}^4}\int dk k^2 w_k\frac{\left[2w_k^2+m_\chi^2\tilde{a}^2\right]}{w_k^2}\frac{f_A\alpha_0^A m_A}{8M_{\rm Pl}^2\tilde{a}^{3/2}i}\\&\qquad\times 2i\Im\left[\beta_k^*\left(e^{im_A\psi_{k,1}^A(\eta')}-e^{im_A\psi_{k,2}^A(\eta')}\right)\right],\end{aligned}\tag{77}$$

where $\langle\rangle$ denotes taking the average over several oscillations. However, we can see that, due to the oscillatory nature of the integrand, this average will only give a non-zero result if η coincides with a stationary point of the phase of one of the terms in the integrand. As there are no stationary points of $\psi_{k,2}^A(\eta)$ for physical values of $\tilde{a}$, the only terms giving a non-zero contribution are those containing the phase $\psi_{k,1}^A(\eta)$, namely we find

$$\begin{aligned}\left\langle\frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\chi+3\tilde{p}_\chi)\right\rangle &= \int dk\sum_A\delta(\eta-\eta_k^A)\int_{\eta_k^A-T}^{\eta_k^A+T}d\eta'\frac{4\pi}{(2\pi)^3\tilde{a}^4}k^2 w_k\frac{\left[2w_k^2+m_\chi^2\tilde{a}^2\right]}{w_k^2}\frac{f_A\alpha_0^A m_A}{8M_{\rm Pl}^2\tilde{a}^{3/2}i}\\&\qquad\times 2i\Im\left[\beta_k^* e^{im_A\psi_{k,1}^A(\eta')}\right].\end{aligned}\tag{78}$$

Seeing as the integral over η' is centred on the stationary point for each A, we can take $T\to\infty$, as contributions away from the stationary point will average to zero. On making the stationary phase approximation, and after a little manipulation we eventually find

$$\begin{aligned}\left\langle\frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\chi+3\tilde{p}_\chi)\right\rangle &= \sum_{A,B}\frac{4\pi}{(2\pi)^3\tilde{a}^5(\eta)}\int dk\delta(\eta-\eta_k^A)k^2 w_k(\eta_k^A)\beta_k^A\beta_k^{B*}\\&\qquad\times 2\cos\left(m_A\psi_{k,1}^A(\eta_k^A)-m_B\psi_{k,1}^B(\eta_k^B)+(s_k^A-s_k^B)\pi/4\right)\Theta(\eta_k^A-\eta_k^B)\Theta(\eta_k^B-\eta_0),\end{aligned}\tag{79}$$

where we have used (69) and (70). If we arrange that $m_B>m_A$ for $B>A$, meaning that $\Theta(\eta_k^A-\eta_k^B)=1$

only for $B > A$, then (79) can be written as

$$\left\langle \frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\chi + 3\tilde{p}_\chi)\right\rangle = \sum_A \frac{4\pi}{(2\pi)^3\tilde{a}^5(\eta)}\int dk\delta(\eta-\eta_k^A)k^2 w_k(\eta_k^A)|\beta_k^A|^2 + \sum_{A,B>A}\frac{4\pi}{(2\pi)^3\tilde{a}^5(\eta)}\int dk\delta(\eta-\eta_k^A)k^2 w_k(\eta_k^A)\beta_k^A\beta_k^{B*} \times 2\cos\left(m_A\psi_{k,1}^A(\eta_k^A) - m_B\psi_{k,1}^B(\eta_k^B) + (s_k^A - s_k^B)\pi/4\right)\Theta(\eta_k^B-\eta_0). \tag{80}$$

The delta function in η can then be expressed as a delta function in k by using the fact that $\delta(\eta-\eta_k^A) = |\psi_{k,1}^{A\prime\prime}(\eta_k^{\hat{A}})|\delta(\psi_{k,1}^{\hat{A}\prime}(\eta))$ in combination with relation (71), and we find

$$\delta(\eta-\eta_k^A) = \frac{m_A^2}{4\mu_{\hat{A}}}|\psi_{k,1}^{\hat{A}\prime\prime}(\eta_k^{\hat{A}})|\delta(k-\tilde{a}(\eta)\mu_{\hat{A}}), \qquad \text{where} \qquad \mu_A \simeq \frac{m_A}{2}\left(1-\frac{4m_\chi^2}{m_{\hat{A}}^2}\right)^{1/2}. \tag{81}$$

If we consider only the diagonal contributions to the double summation in (80), i.e. the terms on the first line, and assume that off-diagonal terms average to zero due to the cosine function, then we find

$$\left\langle \frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\chi + 3\tilde{p}_\chi)\right\rangle = \sum_A \frac{1}{\tilde{a}^3(\eta)}\frac{1}{256M_{\rm Pl}^4\pi}\frac{\left[m_A f_A\alpha_0^A(2m_\chi^2+m_A^2)\right]^2}{m_A}\left(1-\frac{4m_\chi^2}{m_A^2}\right)^{1/2}, \tag{82}$$

and by using the fact that $\tilde{\rho}_A = m_{\hat{A}}^2(\alpha_0^A)^2/2\tilde{a}^3$, we can then express this as

$$\left\langle \frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\chi + 3\tilde{p}_\chi)\right\rangle = \sum_A \frac{1}{128M_{\rm Pl}^4\pi}\frac{\left[f_A(2m_\chi^2+m_A^2)\right]^2}{m_A}\left(1-\frac{4m_\chi^2}{m_A^2}\right)^{1/2}\tilde{\rho}_A \equiv \sum_A \tilde{\Gamma}_{\alpha^A\to\chi\chi}\tilde{\rho}_A, \tag{83}$$

i.e. we have found

$$\tilde{\Gamma}_{\alpha^A\to\chi\chi} = \frac{\left[f_A(2m_\chi^2+m_{\hat{A}}^2)\right]^2}{128\pi M_{\rm Pl}^4 m_{\hat{A}}}\left(1-\frac{4m_\chi^2}{m_{\hat{A}}^2}\right)^{1/2}, \tag{84}$$

which is in agreement with (36).

The production of fermions

In considering the case for fermions we follow closely the analyses of [48–51]. As many aspects of the calculation are similar to the bosonic case, we defer details to Appendix B.

From the fermionic action in (2) we obtain the Dirac equation

$$(\gamma^\mu(x)D_\mu + m_\psi)\,\psi = 0. \tag{85}$$

Specialising to the case of a FLRW metric, and taking $e_a^0 = \delta_a^0$ and $e_a^i = \delta_a^i/a(t)$, we have

$$\gamma^0(x) = \gamma^0, \qquad \gamma^i(x) = \frac{\gamma^i}{a(t)}, \qquad \Gamma_0 = 0 \quad \text{and} \quad \Gamma_i = \frac{\dot{a}}{2}\gamma^0\gamma^i. \tag{86}$$

If we introduce conformal time, and also define $\Psi = a^{3/2}\psi$, the Dirac equation can be written as

$$(\gamma^a\partial_a + am_\psi)\,\Psi = 0, \tag{87}$$

where in this equation $\partial_0 = \partial_\eta$. Correspondingly, the action and Hamiltonian can be written as

$$S_\Psi = -\int d\eta d^3x\overline{\Psi}\,(\gamma^a\partial_a + am_\psi)\,\Psi \qquad \text{and} \qquad \mathrm{H}_\Psi = -\int d^3x\overline{\Psi}\gamma^0\partial_\eta\Psi = i\int d^3x\Psi^\dagger\partial_\eta\Psi. \tag{88}$$

The space of solutions is endowed with a conserved scalar product, and in the FLRW case it reduces to

$$(\Psi_1, \Psi_2) = \int d^3x \Psi_1^\dagger \Psi_2. \tag{89}$$

Given one solution to the Dirac equation, $U_r(\vec{k}, x)$, one can show that the charge conjugate $V_r(\vec{k}, x) = C\overline{U}_r^T(\vec{k}, x) = \gamma^2 U_r^*(\vec{k}, x)$, where $C = \gamma^2\beta$, is also a solution. Note that the subscript r labels the spin. We can then construct a basis of the solution space out of $U_r(\vec{k}, x)$ and $V_r(\vec{k}, x)$, and further require that the basis be orthonormal with respect to the above scalar product. As such, a general solution can be decomposed as

$$\Psi(x) = \sum_r \int d^3k \left(a_r(\vec{k}) U_r(\vec{k}, x) + b_r^\dagger(\vec{k}) V_r(\vec{k}, x) \right), \tag{90}$$

where $a_r(\vec{k})$ and $b_r^\dagger(\vec{k})$ now correspond to annihilation and creation operators satisfying the anti-commutation relations

$$\left\{ a_r(\vec{k}), a_s^\dagger(\vec{q}) \right\} = \delta_{rs}\delta^{(3)}(\vec{k} - \vec{q}) \qquad \text{and} \qquad \left\{ b_r(\vec{k}), b_s^\dagger(\vec{q}) \right\} = \delta_{rs}\delta^{(3)}(\vec{k} - \vec{q}), \tag{91}$$

with all other commutators vanishing. We next decompose the solutions $U_r(\vec{k}, x)$ as

$$U_r(\vec{k}, x) = \frac{1}{(2\pi)^{3/2}} \begin{pmatrix} u_{\mathcal{A}}(k, \eta) h_r(\hat{k}) \\ r u_{\mathcal{B}}(k, \eta) h_r(\hat{k}) \end{pmatrix} e^{i\vec{k}\cdot\vec{x}}, \tag{92}$$

where $\hat{k} = \vec{k}/k$ and $h_r(\hat{k})$ are the eigenvectors of the helicity operator

$$\hat{k} \cdot \vec{\sigma} h_r(\hat{k}) = r h_r(\hat{k}), \qquad r = \pm 1, \tag{93}$$

which are chosen to satisfy $h_r^\dagger(\hat{k}) h_s(\hat{k}) = \delta_{rs}$. For the choice of $h_r(\hat{k})$ made in Appendix B, we then find

$$V_r(\vec{k}, x) = \gamma^2 U_r^*(\vec{k}, x) = \frac{e^{i\phi_{\hat{k}}}}{(2\pi)^{3/2}} \begin{pmatrix} -u_{\mathcal{B}}^*(k, \eta) h_r(-\hat{k}) \\ r u_{\mathcal{A}}^*(k, \eta) h_r(-\hat{k}) \end{pmatrix} e^{-i\vec{k}\cdot\vec{x}}. \tag{94}$$

Imposing the orthonormality conditions dictates that

$$|u_{\mathcal{A}}(k, \eta)|^2 + |u_{\mathcal{B}}(k, \eta)|^2 = 1, \tag{95}$$

and the Dirac equation now takes the form

$$i\partial_\eta \begin{pmatrix} u_{\mathcal{A}}(k, \eta) \\ u_{\mathcal{B}}(k, \eta) \end{pmatrix} = \begin{pmatrix} am_\psi & k \\ k & -am_\psi \end{pmatrix} \begin{pmatrix} u_{\mathcal{A}}(k, \eta) \\ u_{\mathcal{B}}(k, \eta) \end{pmatrix}, \tag{96}$$

which can be decoupled to

$$u''_{\mathcal{A},\mathcal{B}}(k, \eta) = -\left[k^2 + a^2 m_\psi^2 \pm i(am_\psi)' \right] u_{\mathcal{A},\mathcal{B}}(k, \eta), \tag{97}$$

These two equations are now of the same form as Eq. (50) for the boson mode functions. As such, the procedure from here onwards is very similar to the bosonic case. In analogy with the the bosonic case, we expand $u_{\mathcal{A}}(k, \eta)$ and $u_{\mathcal{B}}(k, \eta)$ in terms of positive and negative frequency functions as

$$u_{\mathcal{A}}(k, \eta) = \mathcal{A}_k(\eta) \sqrt{\frac{w_k + am_\psi}{2w_k}} e^{-i\int w_k d\eta'} - \mathcal{B}_k(\eta) \sqrt{\frac{w_k - am_\psi}{2w_k}} e^{i\int w_k d\eta'}, \tag{98}$$

$$u_{\mathcal{B}}(k, \eta) = \mathcal{A}_k(\eta) \sqrt{\frac{w_k - am_\psi}{2w_k}} e^{-i\int w_k d\eta'} + \mathcal{B}_k(\eta) \sqrt{\frac{w_k + am_\psi}{2w_k}} e^{i\int w_k d\eta'}, \tag{99}$$

where $w_k^2 = k^2 + a^2 m_\psi^2$. With this decomposition we find that $\mathcal{A}_k(\eta)$ and $\mathcal{B}_k(\eta)$ must satisfy the normalisation condition $|\mathcal{A}_k(\eta)|^2 + |\mathcal{B}_k(\eta)|^2 = 1$ and the evolution equations

$$\mathcal{A}'_k(\eta) = -\frac{k(am_\psi)'}{2w_k^2} e^{2i\int w_k d\eta'} \mathcal{B}_k(\eta), \tag{100}$$

$$\mathcal{B}'_k(\eta) = \frac{k(am_\psi)'}{2w_k^2} e^{-2i\int w_k d\eta'} \mathcal{A}_k(\eta). \tag{101}$$

Assuming that at some time in the past $\mathcal{B}_k(\eta_0) = 0$, the above mode functions then coincide with the flat-space mode functions and the Hamiltonian is diagonal. As $\mathcal{B}_k(\eta)$ evolve away from zero, however, the Hamiltonian is no longer diagonal, instead taking the form [49–51]

$$\mathrm{H}_\Psi = \sum_r \int d^3k \Big[w_k \left(|\mathcal{A}_k(\eta)|^2 - |\mathcal{B}_k(\eta)|^2 \right) \left(a_r^\dagger(\vec{k}) a_r(\vec{k}) - b_r(\vec{k}) b_r^\dagger(\vec{k}) \right) \tag{102}$$

$$-2\mathcal{A}_k(\eta)\mathcal{B}_k(\eta) w_k e^{-i\phi_{-\hat{k}}} b_r(-\vec{k}) a_r(\vec{k}) - 2\mathcal{A}_k^*(\eta)\mathcal{B}_k^*(\eta) w_k e^{i\phi_{-\hat{k}}} a^\dagger(\vec{k}) b_r^\dagger(-\vec{k}) \Big].$$

In order to diagonalise the Hamiltonian one can make a Bogoliubov transformation, defining

$$\hat{a}_r(\vec{k},\eta) = \mathcal{A}_k(\eta) a_r(\vec{k}) - \mathcal{B}_k^*(\eta) e^{i\phi_{-\hat{k}}} b_r^\dagger(-\vec{k}), \qquad \hat{b}_r^\dagger(\vec{k},\eta) = \mathcal{B}_k(\eta) e^{-i\phi_{\hat{k}}} a_r(-\vec{k}) + \mathcal{A}_k^*(\eta) b_r^\dagger(k). \tag{103}$$

One then finds that the number operator associated with the new basis is given as $\langle 0|\hat{a}_r^\dagger(\vec{k},\eta)\hat{a}_r(\vec{k},\eta)|0\rangle = \langle 0|\hat{b}_r^\dagger(\vec{k},\eta)\hat{b}_r(\vec{k},\eta)|0\rangle = |\mathcal{B}_k(\eta)|^2$. We must therefore determine $\mathcal{B}_k(\eta)$ if we wish to determine the number of particles created. Looking at (101) we see that the form of the equation we need to solve for $\mathcal{B}_k(\eta)$ is almost identical to that for $\beta_k(\eta)$ that we solved in the case of the bosonic field.[4] As such, we defer details of the calculation to Appendix B, stating only the main results here.

First, on using the stationary phase approximation we find that $\mathcal{B}_k(\eta)$ is given as

$$\mathcal{B}_k(\eta) = \sum_A \mathcal{B}_k^A \exp\left[i m_A \psi_{k,1}^A(\eta_k^A) + i s_k^A \pi/4 \right] \Theta(\eta - \eta_k^A)\Theta(\eta_k^A - \eta_0), \tag{104}$$

$$\mathcal{B}_k^A = \frac{k f_{\hat{A}} m_{\hat{A}} \alpha_0^A m_\psi}{8 M_{\mathrm{Pl}}^2 i w_k^2(\eta_k^{\hat{A}})} \sqrt{\frac{2\pi \tilde{a}(\eta_k^{\hat{A}})}{m_{\hat{A}} |\psi_{k,1}^{\hat{A}\prime\prime}(\eta_k^{\hat{A}})|}}, \tag{105}$$

where $\psi_{k,1}^A(\eta)$ is still as defined in (67) but with $w_k^2 = k^2 + a^2 m_\psi^2$. As in the bosonic case, η_k^A is the time at which $\psi_{k,1}^{A\prime}(\eta) = 0$ is satisfied, i.e. the time at which the phase is stationary, and s_k^A is the sign of $\psi_{k,1}^{A\prime\prime}(\eta_k^{\hat{A}})$. Then, as with the scalar case, we wish to determine the quantity $(df/d\tilde{t})\tilde{T}^{\tilde{\psi}}/(2f)$, which corresponds to the right-hand side of the continuity equation for the energy-momentum associated with $\tilde{\psi}$ in the Einstein frame, i.e. corresponds to the particle production term. Taking $\langle 0|\tilde{T}^{(\tilde{\psi})\mu}{}_\nu|0\rangle = \mathrm{diag}(-\tilde{\rho}_\psi,\ \tilde{p}_\psi,\ \tilde{p}_\psi,\ \tilde{p}_\psi)$, we find that to first order in $\mathcal{B}_k(\eta)$

$$\tilde{T}^{(\tilde{\psi})} = -\tilde{\rho}_\psi + 3\tilde{p}_\psi \simeq -\frac{4}{(2\pi)^3 \tilde{a}^4} \int d^3k \frac{k \tilde{a} m_\psi}{w_k \Omega} \Re\left(\mathcal{A}_k(\eta) \mathcal{B}_k^*(\eta) e^{-2i\int_{-\infty}^{\eta} w_k d\eta'} \right). \tag{106}$$

Proceeding in exactly the same way as for the bosonic field in the previous subsection, we arrive at

$$\left\langle \frac{1}{2f}\frac{df}{d\tilde{t}}\left(-\tilde{\rho}_\psi + 3\tilde{p}_\psi\right) \right\rangle = \sum_A \tilde{\rho}_A \frac{(f_A)^2 m_\psi^2 m_A}{32\pi M_{\mathrm{Pl}}^4} \left(1 - \frac{4m_\psi^2}{m_A^2} \right)^{3/2}, \tag{107}$$

from which we deduce

$$\tilde{\Gamma}_{\alpha^A \to \overline{\psi}\psi} = \frac{(f_A)^2 m_\psi^2 m_{\hat{A}}}{32\pi M_{\mathrm{Pl}}^4} \left(1 - \frac{4m_\psi^2}{m_{\hat{A}}^2} \right)^{3/2}, \tag{108}$$

which is in agreement with (36).

Energy-momentum tensors and their (non-)conservation

In the analysis of the preceding two subsections we considered the continuity equation for the matter energy-momentum tensor in the Einstein frame. Our reason for doing so was the transparent interpretation:

[4] Indeed, if we had considered a conformally coupled field instead of a minimally coupled one, the evolution equation for the Bogoliubov coefficients would be the same up to a factor of $k/(ma)$ [46].

the non-conservation of the matter energy-momentum tensor in the Einstein frame is a result of the explicit interaction terms that give rise to particle production. To close this section we consider the continuity equations for the other energy-momentum tensors.

First let us consider the energy-momentum tensor of the oscillating fields in the Einstein frame. Combining the results of the previous two subsections we have

$$\tilde{\nabla}_\mu \tilde{T}^{(m)\mu}{}_0 = -\sum_A \tilde{\Gamma}_A \tilde{\rho}_A. \tag{109}$$

As the total energy-momentum tensor must be conserved, this implies that $\tilde{\nabla}_\mu \tilde{T}^{(\phi)\mu}{}_0 = \sum_A \tilde{\Gamma}_A \tilde{\rho}_A$, where $\tilde{T}^{(\phi)\mu}{}_\nu = \sum_A \mathrm{diag}(-\tilde{\rho}_A, \tilde{p}_A, \tilde{p}_A, \tilde{p}_A)$, with $\tilde{\rho}_A$ as given in (43) and

$$\tilde{p}_A = \frac{1}{2}\left(\left(\frac{d\alpha^A}{d\tilde{t}}\right)^2 - m_{\hat{A}}^2 (\alpha^A)^2\right). \tag{110}$$

Averaging over several oscillations we have $\langle \tilde{p}_A \rangle = 0$, so that we obtain the expected continuity equation (44). As previously mentioned, the interpretation of this standard result is intuitive – the energy density of the oscillating fields decays both as a result of the Hubble expansion and the decay into matter. Under the instant decay approximation, we assume that reheating ends once the decay rate "catches up" with the Hubble expansion, i.e. when $\tilde{\Gamma}_\alpha = 3\tilde{H}$. This then allows us to determine the reheating temperature in terms of $\tilde{\Gamma}_\alpha$, and thus put constraints on model parameters such as f_A and m_A.

Next we consider the matter energy-momentum tensor in the Jordan frame, which we know to be covariantly conserved – recall (16). At first glance this property would seem to be at odds with the fact that we have particle production. However, in the Jordan frame the particle production is interpreted as being due to the oscillatory nature of the Hubble rate, and the term on the right-hand side of (73), for example, becomes part of the Hubble dilution term on the left-hand side of the continuity equation in the Jordan frame. Namely, using $\rho_\chi = \Omega^4 \tilde{\rho}_\chi$, $p_\chi = \Omega^4 \tilde{p}_\chi$ and the relations given in (22), (73) can be re-written as the standard continuity equation

$$\frac{1}{a^4}\frac{d}{dt}(a^4 \rho_\chi) + H(-\rho_\chi + 3p_\chi) = 0. \tag{111}$$

Note that whilst we have considered the bosonic field as an example, the same is also true for any matter field.

Finally we consider the energy-momentum tensor for the oscillating fields in the Jordan frame. In the Jordan frame there is some ambiguity as to how we might like to define the energy-momentum tensor of the inflaton fields, and the relation between the Jordan and Einstein frame inflaton energy-momentum tensors is not just a simple factor of Ω^2, as it is for the matter energy-momentum tensors. As commented in Appendix A, Einstein's equations in the Jordan frame can be recast into the standard form if we define the effective energy-momentum tensor given in (A10). We then choose to define $T^{(\phi,\mathrm{eff})}_{\mu\nu}$ such that

$$T^{(\mathrm{eff})}_{\mu\nu} = T^{(\phi,\mathrm{eff})}_{\mu\nu} + \frac{M_{\mathrm{Pl}}^2}{f} T^{(m)}_{\mu\nu}, \tag{112}$$

i.e. we have

$$T^{(\phi,\mathrm{eff})}_{\mu\nu} = \frac{M_{\mathrm{Pl}}^2}{f}\left[T^{(\phi)}_{\mu\nu} + \nabla_\mu\nabla_\nu f - g_{\mu\nu}\Box f\right]. \tag{113}$$

As such, we see that despite the fact that $T^{(m)}_{\mu\nu}$ is covariantly conserved, $T^{(\phi,\mathrm{eff})}_{\mu\nu}$ is not, with

$$\nabla_\mu T^{(\phi,\mathrm{eff})\mu}{}_\nu = \frac{M_{\mathrm{Pl}}^2}{f^2} T^{(m)\mu}{}_\nu \nabla_\mu f. \tag{114}$$

In a FLRW background we explicitly have

$$\rho_\phi^{\mathrm{eff}} = \frac{M_{\mathrm{Pl}}^2}{f}\left[\frac{1}{2}h_{ab}\dot{\phi}^a\dot{\phi}^b + V - 3H\dot{f}\right] = \frac{f}{M_{\mathrm{Pl}}^2}\tilde{\rho}_\phi - 3\tilde{H}\frac{df}{d\tilde{t}} + \frac{3}{4f}\left(\frac{df}{d\tilde{t}}\right)^2, \tag{115}$$

$$p_\phi^{\mathrm{eff}} = \frac{M_{\mathrm{Pl}}^2}{f}\left[\frac{1}{2}h_{ab}\dot{\phi}^a\dot{\phi}^b - V + \ddot{f} + 2H\dot{f}\right] = \frac{f}{M_{\mathrm{Pl}}^2}\tilde{p}_\phi + \frac{d^2 f}{d\tilde{t}^2} + 2\tilde{H}\frac{df}{d\tilde{t}} - \frac{5}{4f}\left(\frac{df}{d\tilde{t}}\right)^2, \tag{116}$$

and the continuity equation

$$\dot{\rho}_\phi^{\rm eff} + 3H(\rho_\phi^{\rm eff} + p_\phi^{\rm eff}) = \frac{M_{\rm Pl}^2}{f^2}\dot{f}\rho^{(m)}, \tag{117}$$

where $\rho_\phi^{\rm eff} = -T^{(\phi,{\rm eff})\,0}{}_0$, $p_\phi^{\rm eff} = T^{(\phi,{\rm eff})\,i}{}_i/3$, $\tilde{\rho}_\phi = -\tilde{T}^{(\phi)0}{}_0$, $\tilde{p}_\phi = \tilde{T}^{(\phi)i}{}_i/3$ and $\rho^{(m)} = -T^{(m)0}{}_0$. The physical interpretation of this last equation is less clear than that of (44) in the Einstein frame. However, one can assume that reheating completes when $\rho^{(m)} \approx 3(\rho_\phi^{\rm eff} + p_\phi^{\rm eff})/2$, where we have used the fact that $\dot{f}/f \approx -2H$, as can be seen from (22).

In the above analysis we have derived conditions for instant reheating in both the Jordan and Einstein frames. However, we note that imposing instant reheating in this class of models gives rise to issues regarding the discontinuity of H or $\tilde{H}$, which results from the assumption that $f \to M_{\rm Pl}^2$ instantaneously at the time of reheating [13]. To avoid this issue one must therefore solve the continuity equations dynamically.

IV. SUMMARY AND CONCLUSIONS

The high-precision nature of current CMB data dictates that reheating dynamics must be taken into account when trying to constrain different models of inflation. Given the recent interest in inflation models containing a non-minimal coupling to gravity and potentially multiple scalar fields, in this paper we have revisited the process of gravitational reheating that is inherent to this class of model. Our formulation allows for multiple, non-minimally coupled inflaton fields endowed with a non-flat field-space metric, and it is assumed that these fields are not directly coupled to matter.

At the level of the background dynamics, we saw that the oscillation of the inflaton fields about their vacuum expectation values gives rise to matter-dominated-like evolution of the Hubble rate in the Einstein frame, as in elementary reheating scenarios. In the Jordan frame, however, this matter-dominated-like evolution is modulated by an oscillatory component, and it is this oscillatory part that gives rise to the gravitational particle production of minimally-coupled matter, i.e. gravitational reheating. When interpreted in the Einstein frame the gravitational reheating does not result from the oscillatory nature of the Hubble rate, but instead from the explicit interaction terms between the inflaton sector and ordinary matter that are induced by the conformal transformation.

In order to calculate the rate of particle production we used the method of QFT in a classical background, which requires the calculation of Bogoliubov coefficients. Although this was not entirely necessary for the perturbative reheating regime considered, the advantage is that much of the discussion will also carry over to the resonant preheating regime, where the perturbative flat-space QFT calculations are no longer applicable. Taking appropriate limits, we were able to confirm agreement between the Bogoliubov and perturbative QFT approaches, including kinematic suppression factors. Despite the difference in interpretation between the Jordan and Einstein frames, we saw that the calculation of the Bogoliubov coefficients associated with particle production was independent of the frame in which we started. This resulted from the fact that the canonically normalised quantum fields one naturally defines in the two frames are identical.

To finish, let us mention one possible extension of the framework developed here. In analysing the dynamics of the oscillating inflaton fields at the end of inflation we made use of the mass eigen-basis of the Einstein frame potential. We implicitly made the assumption that all of the fields begin oscillating about their vacuum expectation values at approximately the same time, with $m_A \sim m_B \gg \tilde{H}$ for all A and B. Such an approximation, however, may not be valid. Generally we might expect there to be a wide range of field masses, and that different fields therefore begin to oscillate and decay at different times. In the case that heavier fields are present, which start to oscillate and decay much earlier, it is perhaps reasonable to assume that the resulting decay products are diluted by inflation – which continues to be driven by the lighter fields – and are therefore negligible. However, in the case that lighter spectator fields are present, which do not oscillate and decay until much later, we might expect them to play a significant role. In general we would expect the field space metric h_{ab}, potential $V(\phi)$ and non-minimal coupling function $f(\phi)$ to all depend on these spectator fields. Consequently, quantities such as $S_{ab}|_{\rm vev}$ and $\tilde{V}_{ab}|_{\rm vev}$, which we took to be constants in the analysis of Sec. III A, would all become functions of the spectator fields. Ultimately, this would then lead to a spectator-field dependence of $\tilde{\Gamma}_{\alpha^A\to\chi\chi}$ and $\tilde{\Gamma}_{\alpha^A\to\overline{\psi}\psi}$, through their dependence on m_A and f_A, which would in turn give rise to a modulated reheating scenario. We leave further consideration of this scenario to future work.

ACKNOWLEDGMENTS

This work was supported by Japan Society for the Promotion of Science (JSPS) Research Fellowship for Young Scientists No. 269337 (Y.W.) and JSPS Grant-in-Aid for Scientific Research (B) No. 23340058 (J.W.). Y.W. acknowledges support from the Munich Institute for Astro- and Particle Physics (MIAPP) of the Deutsche Forschungsgemeinschaft (DFG) cluster of excellence "Origin and Structure of the Universe."

Appendix A: Energy-momentum tensors and their (non-)conservation

In this appendix we review in more detail the relation between energy-momentum tensors defined in the Jordan and Einstein frames, including determining whether or not they are covariantly conserved. We follow closely [52–54], simply generalising to the multi-field case.

Covariant conservation of the matter energy-momentum tensor in the Jordan frame

Let us start by showing that the energy-momentum tensor for matter in the Jordan frame is covariantly conserved, despite the presence of the non-minimal coupling. In Sec. II B we derived the Einstein equations in the Jordan frame as

$$G_{\mu\nu} = \frac{1}{f}\left[T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu} + \nabla_\mu\nabla_\nu f - g_{\mu\nu}\Box f\right], \tag{A1}$$

where, using the definition

$$T^{(i)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(i)}\right)}{\delta g^{\mu\nu}}, \tag{A2}$$

we have

$$T^{(\phi)}_{\mu\nu} = h_{ab}\nabla_\mu\phi^a\nabla_\nu\phi^b - g_{\mu\nu}\left(\frac{1}{2}h_{ab}g^{\rho\sigma}\nabla_\rho\phi^a\nabla_\sigma\phi^b + V\right). \tag{A3}$$

We also gave the equations of motion for the fields as

$$h_{ab}\Box\phi^b + \Gamma_{bc|a}g^{\mu\nu}\nabla_\mu\phi^b\nabla_\nu\phi^c - V_a + f_a R = 0, \tag{A4}$$

where $\Gamma_{ab|c} = h_{cd}\Gamma^d_{ab}$ and Γ^a_{bc} is the Christoffel connection associated with the field-space metric h_{ab}. Taking the covariant divergence of (A1), and using the Bianchi identity, we find

$$\nabla^\mu G_{\mu\nu} \equiv 0 = -\frac{\nabla^\mu f}{f}G_{\mu\nu} + \frac{1}{f}\left[\nabla^\mu T^{(\phi)}_{\mu\nu} + \nabla^\mu T^{(m)}_{\mu\nu} + \Box\nabla_\nu f - \nabla_\nu\Box f\right]. \tag{A5}$$

Similarly, taking the covariant derivative of (A3) and using the equations of motion (A4), we also find

$$\nabla^\mu T^{(\phi)}_{\mu\nu} = -R\nabla_\nu f. \tag{A6}$$

Substituting this result into (A5), and recalling $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, we arrive at

$$\nabla^\mu G_{\mu\nu} \equiv 0 = -\frac{\nabla^\mu f}{f}R_{\mu\nu} + \frac{1}{f}\left[\nabla^\mu T^{(m)}_{\mu\nu} + \Box\nabla_\nu f - \nabla_\nu\Box f\right]. \tag{A7}$$

However, from the definition of the Riemann tensor we have

$$\Box\nabla_\nu f - \nabla_\nu\Box f = R_{\nu\mu}\nabla^\mu f, \tag{A8}$$

which clearly then leaves us with

$$\nabla^\mu G_{\mu\nu} \equiv 0 = \frac{1}{f}\nabla^\mu T^{(m)}_{\mu\nu}, \tag{A9}$$

i.e. we have recovered the fact that the matter energy-momentum tensor in the Jordan frame is covariantly conserved.

One could also consider an effective energy-momentum tensor defined by (A1) as

$$T^{(\mathrm{eff})}_{\mu\nu} = \frac{M_{\mathrm{Pl}}^2}{f}\left[T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu} + \nabla_\mu\nabla_\nu f - g_{\mu\nu}\Box f\right], \tag{A10}$$

such that Einstein's equations take the standard form $G_{\mu\nu} = T^{(\mathrm{eff})}_{\mu\nu}/M_{\mathrm{Pl}}^2$. This effective energy-momentum tensor is of course covariantly conserved as a result of the Bianchi identity.

Energy-momentum tensors in the Einstein frame

Turning to the Einstein frame, if we assume that $\mathcal{L}^{(m)}$ only depends on $g_{\mu\nu}$ and not its derivatives, then we can write

$$T^{(m)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(m)}\right)}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(m)}\right)}{\delta \tilde{g}^{\rho\sigma}}\frac{\partial \tilde{g}^{\rho\sigma}}{\partial g^{\mu\nu}}. \tag{A11}$$

Under the conformal transformation we have $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $\tilde{g}^{\mu\nu} = g^{\mu\nu}/\Omega^2$ and $\sqrt{-\tilde{g}} = \Omega^4\sqrt{-g}$, which we can substitute into (A11) to find

$$T^{(m)}_{\mu\nu} = \Omega^2 \tilde{T}^{(m)}_{\mu\nu}. \tag{A12}$$

Recall that $\Omega^2 = f(\phi)/M_{\mathrm{Pl}}^2$. Taking the covariant divergence of this (with respect to the Jordan frame metric) we have

$$\nabla^\mu T^{(m)}_{\mu\nu} = \Omega^4 \tilde{g}^{\mu\alpha}\nabla_\alpha \tilde{T}^{(m)}_{\mu\nu} + 2\Omega^3\tilde{g}^{\mu\alpha}\Omega_\alpha\tilde{T}^{(m)}_{\mu\nu}. \tag{A13}$$

We now need to use the relation between covariant derivatives as defined with respect to $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. The relation between the Christoffel symbols is given as

$$\Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma} - \frac{1}{\Omega}\left(\delta^\alpha_\beta\Omega_\gamma + \delta^\alpha_\gamma\Omega_\beta - \tilde{g}^{\alpha\rho}\tilde{g}_{\gamma\beta}\Omega_\rho\right), \tag{A14}$$

and on substituting this result into (A13) we find

$$\nabla^\mu T^{(m)}_{\mu\nu} = 0 = \Omega^4\tilde{\nabla}^\mu\tilde{T}^{(m)}_{\mu\nu} + \tilde{T}^{(m)}\Omega^3\Omega_\nu \qquad \Rightarrow \qquad \tilde{\nabla}^\mu\tilde{T}^{(m)}_{\mu\nu} = -\tilde{T}^{(m)}\frac{\Omega_\nu}{\Omega}, \tag{A15}$$

where $\tilde{T}^{(m)}$ is the trace of the matter energy-momentum tensor. Thus, we see that even if $T^{(m)}_{\mu\nu}$ is covariantly conserved, in general $\tilde{T}^{(m)}_{\mu\nu}$ is not. It will, however, be conserved if $\tilde{T}^{(m)} = 0$, which is the case for radiation-like matter.

Given that in the Einstein frame we have

$$\tilde{\nabla}^\mu\tilde{G}_{\mu\nu} \equiv 0 = \tilde{\nabla}^\mu\left(\tilde{T}^{(\phi)}_{\mu\nu} + \tilde{T}^{(m)}_{\mu\nu}\right), \tag{A16}$$

the non-conservation of $\tilde{T}^{(m)}_{\mu\nu}$ implies a non-conservation of $\tilde{T}^{(\phi)}_{\mu\nu}$. Let us try to determine this explicitly.

When we try to calculate the equations of motion for the scalar fields in the Einstein frame, we need to correctly take into account the dependence of $\sqrt{-g}\mathcal{L}^{(m)}$ on ϕ^a that results from the conformal transformation. However, as the only dependence of $\sqrt{-g}\mathcal{L}^{(m)}$ on ϕ^a comes from the conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, if we once again assume that $\sqrt{-g}\mathcal{L}^{(m)}$ only depends on $g_{\mu\nu}$ and not its derivatives, then we can simply use the rules of partial differentiation to obtain

$$\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(m)}\right)}{\delta\phi^a} = \frac{\delta\left(\sqrt{-g}\mathcal{L}^{(m)}\right)}{\delta g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial\phi^a} = 2\Omega\Omega_a\tilde{g}^{\mu\nu}\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(m)}\right)}{\delta g^{\mu\nu}} = 2\frac{\Omega_a}{\Omega}\tilde{g}^{\mu\nu}\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(m)}\right)}{\delta \tilde{g}^{\mu\nu}} = -\frac{\Omega_a}{\Omega}\sqrt{-\tilde{g}}\tilde{T}^{(m)}. \tag{A17}$$

As such, the equations of motion for the scalar fields become

$$-S_{ab}\tilde{\Box}\phi^b -^{(S)}\Gamma_{bc|a}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi^b\tilde{\nabla}_\nu\phi^c + \tilde{V}_{,a} + \frac{\Omega_a}{\Omega}\tilde{T}^{(m)} = 0, \tag{A18}$$

where ${}^{(S)}\Gamma_{bc|a} = S_{ad}{}^{(S)}\Gamma^d_{bc}$ and ${}^{(S)}\Gamma^d_{bc}$ is the Christoffel connection associated with S_{ab}. As given in the main text, the energy momentum tensor $\tilde{T}^{(\phi)}_{\mu\nu}$ takes the form

$$\tilde{T}^{(\phi)}_{\mu\nu} = S_{ab}\tilde{\nabla}_\mu\phi^a\tilde{\nabla}_\nu\phi^b - \tilde{g}_{\mu\nu}\left(\frac{1}{2}S_{ab}\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\phi^a\tilde{\nabla}_\sigma\phi^b + \tilde{V}\right). \tag{A19}$$

Taking the covariant divergence of this and making use of the equations of motion (A18), we find

$$\tilde{\nabla}^\mu\tilde{T}^{(\phi)}_{\mu\nu} = \frac{\Omega_\nu}{\Omega}\tilde{T}^{(m)}. \tag{A20}$$

This is entirely consistent with what we expected from (A16) and (A15).

Appendix B: Details regarding fermion particle production

In this appendix we give additional details regarding the fermion particle production calculation. As stated in the main text, we follow closely the analyses of [44, 49–51].

Conventions

First let us clarify our conventions. We use the $(-+++)$ sign convention and the Dirac gamma matrix representation

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}, \tag{B1}$$

where σ^i are the Pauli matrices.

Following [51], for the eigenvectors of the helicity operator we use the following spherical-coordinates-based representation

$$h_{+1}(\hat{k}) = \begin{pmatrix} \cos\frac{\theta_{\hat{k}}}{2}e^{-i\phi_{\hat{k}}} \\ \sin\frac{\theta_{\hat{k}}}{2} \end{pmatrix}, \qquad h_{-1}(\hat{k}) = \begin{pmatrix} \sin\frac{\theta_{\hat{k}}}{2}e^{-i\phi_{\hat{k}}} \\ -\cos\frac{\theta_{\hat{k}}}{2} \end{pmatrix}, \tag{B2}$$

where we have chosen the normalisation such that $h^\dagger_r(\hat{k})h_s(\hat{k}) = \delta_{rs}$ is satisfied. With this choice, it is then possible to show that

$$-i\sigma^2 h^*_r(\hat{k}) = -re^{i\phi_{\hat{k}}}h_{-r}(\hat{k}), \tag{B3}$$

which on using $h_r(-\hat{k}) = h_{-r}(\hat{k})$ allows us to find

$$V_r(\vec{k}) = \gamma^2 U^*_r(\vec{k}, x) = \frac{e^{i\phi_{\hat{k}}}}{(2\pi)^{3/2}}\begin{pmatrix} -u^*_{\mathcal{B}}(k,\eta)h_r(-\hat{k}) \\ ru^*_{\mathcal{A}}(k,\eta)h_r(-\hat{k}) \end{pmatrix} e^{-i\vec{k}\cdot\vec{x}}. \tag{B4}$$

Calculation of production rate

Here we give more details regarding the calculation of the production rate for fermions. As with the scalar case, in order to determine the particle production rate we need to evaluate the right-hand side of the continuity equation for the fermion energy-momentum tensor in the Einstein frame. As such, we need to first determine the vacuum expectation values of the components of the energy- momentum tensor, which we denote $\langle 0|\tilde{T}^{(\tilde{\psi})\mu}{}_\nu|0\rangle = \mathrm{diag}(-\tilde{\rho}_\psi,\ \tilde{p}_\psi,\ \tilde{p}_\psi,\ \tilde{p}_\psi)$. In terms of the quantities $\tilde{\rho}_\psi$ and $\tilde{p}_\psi$ the continuity equation takes the form

$$\frac{1}{\tilde{a}^4}\frac{d}{d\tilde{t}}(\tilde{a}^4\tilde{\rho}_\psi) + \tilde{H}(-\tilde{\rho}_\psi + 3\tilde{p}_\psi) = \frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\psi + 3\tilde{p}_\psi). \tag{B5}$$

The explicit form of $\tilde{T}^{(\tilde{\psi})\mu}{}_{\nu}$ for fermions is given in (20), and on writing in terms of the canonically normalised field $\Psi(x)$ given in (90) and taking the vacuum expectation value we find

$$\tilde{\rho}_\psi = \frac{4}{(2\pi)^3\tilde{a}^4}\int d^3k w_k \left(|\mathcal{B}_k(\eta)|^2 - \frac{1}{2}\right), \tag{B6}$$

$$-\tilde{\rho}_\psi + 3\tilde{p}_\psi = \frac{4\tilde{a}m_\psi}{\tilde{a}^4\Omega}\int \frac{d^3k}{(2\pi)^3}\left[\frac{\tilde{a}m_\psi}{w_k\Omega}\left(\frac{1}{2} - |\mathcal{B}_k(\eta)|^2\right) - \frac{k}{w_k}\Re\left(\mathcal{A}_k(\eta)\mathcal{B}_k^*(\eta)e^{-2i\int w_k d\eta'}\right)\right]. \tag{B7}$$

If we are assuming that initially no fermion particles are present, then it is appropriate to consider the perturbative regime where $\mathcal{B}_k(\eta) \ll 1$ and $\mathcal{A}_k(\eta) \simeq 1$. As such, to leading order in $\mathcal{B}_k(\eta)$ and α^A we find

$$\frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\psi + 3\tilde{p}_\psi) \simeq -\frac{1}{2f}\frac{df}{d\tilde{t}}\frac{4(4\pi)}{(2\pi)^3\tilde{a}^4}\int dk k^2 \frac{k\tilde{a}m_\psi}{w_k}\Re\left(\mathcal{B}_k^*(\eta)e^{-2i\int w_k d\eta'}\right). \tag{B8}$$

It is now clear we must solve for $\mathcal{B}_k(\eta)$, i.e. solve Eq. (101). In order to do so, we note that to leading order in α^A we have

$$w_k^2 \simeq k^2 + (\tilde{a}m_\psi)^2, \tag{B9}$$

$$(am_\psi)' \simeq -\tilde{a}m_\psi\frac{f_A\alpha^{A\prime}}{2M_{\rm Pl}^2} + \tilde{a}^2\tilde{H}m_\psi. \tag{B10}$$

Note that, as with the bosonic case, taking expressions to leading order in α^A ensures that we are only considering the tri-linear interaction terms and the perturbative regime appropriate for comparison with the perturbative QFT calculations of Sec. III B. Taking $\mathcal{A}_k(\eta) \to 1$ and integrating (101) we have

$$\mathcal{B}_k(\eta) = \int_{\eta_0}^{\eta} d\eta' \frac{km_\psi}{2w_k^2}\left(\tilde{a}^2\tilde{H}^2 - \tilde{a}\frac{f_A\alpha^{A\prime}}{2M_{\rm Pl}^2}\right)e^{-2i\int_{-\infty}^{\eta'} w_k d\eta''}. \tag{B11}$$

The first term in the brackets is slowly varying, so that its contribution to $\mathcal{B}_k(\eta)$ averages to zero. The second term, however, is highly oscillatory, thus giving a non-zero contribution to $\mathcal{B}_k(\eta)$ that can be calculated using the stationary phase approximation. Explicitly, we have

$$\mathcal{B}_k(\eta) = \sum_A \int_{\eta_0}^{\eta} d\eta' \frac{k\tilde{a}^{1/2}f_A m_A \alpha_0^A m_\psi}{8M_{\rm Pl}^2 i w_k^2}\left(e^{im_A\psi^A_{k,1}(\eta')} - e^{im_A\psi^A_{k,1}(\eta')}\right), \tag{B12}$$

where $\psi^A_{k,1}$ and $\psi^A_{k,2}$ are as given in (67) and (68) but with $w_k^2 = k^2 + a^2m_\psi^2$. The phase $\psi^A_{k,2}$ has no stationary points for physical values of $\tilde{a}$, and therefore the second term in the above expression gives no contribution to $\mathcal{B}_k(\eta)$. The first term involving the phase $\psi^A_{k,1}$, however, does have a stationary point at $\tilde{a} = 2w_k/m_A$. To leading order in α^A, this stationary phase condition gives

$$\frac{k}{\tilde{a}(\eta_k^A)} \simeq \frac{m_A}{2}\left(1 - \frac{4m_\psi^2}{m_{\hat{A}}^2}\right)^{1/2}, \tag{B13}$$

where η_k^A denotes the time at which the condition is satisfied for a given k and m_A. As in the scalar case, this coincides with our expectation from kinematics. In making the stationary phase approximation one also needs the second derivative of the phase at the time η_k^A, which in the fermionic case is given as

$$\psi^{A\prime\prime}_{k,1}(\eta_k^{\hat{A}}) \simeq \tilde{a}^2(\eta_k^{\hat{A}})\tilde{H}(\eta_k^{\hat{A}})\left(1 - \frac{4m_\psi^2}{m_A^2}\right) + \sum_B 2\tilde{a}(\eta_k^{\hat{A}})\frac{m_\psi^2}{m_A^2}\frac{f_B\alpha^{B\prime}(\eta_k^{\hat{A}})}{M_{\rm Pl}^2}. \tag{B14}$$

The final solution for $\mathcal{B}_k(\eta)$ is given in (104).

We next turn to evaluating (B8). On expanding $df/d\tilde{t}$ we obtain

$$\frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\psi + 3\tilde{p}_\psi) = \sum_A \frac{4(4\pi)}{(2\pi)^3\tilde{a}^5}\int dk k^2 w_k \frac{k\tilde{a}^{1/2}f_A\alpha_0^A m_A m_\psi}{8M_{\rm Pl}^2 i w_k^2}2i\Im\left[\mathcal{B}_k^*(\eta)\left(e^{im_A\psi^A_{k,1}(\eta)} - e^{im_A\psi^A_{k,2}(\eta)}\right)\right]. \tag{B15}$$

We see that this is a highly oscillatory function, and as with the scalar case we consider taking an average over several oscillations as

$$\left\langle \frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\psi + 3\tilde{p}_\psi)\right\rangle = \frac{1}{2T}\int_{\eta-T}^{\eta+T} d\eta' \sum_A \frac{4(4\pi)}{(2\pi)^3\tilde{a}^5}\int dk k^2 w_k \frac{k\tilde{a}^{1/2} f_A \alpha_0^A m_A m_\psi}{8M_{\rm Pl}^2 i w_k^2} \times 2i\Im\left[\mathcal{B}_k^*(\eta')\left(e^{im_A\psi_{k,1}^A(\eta')} - e^{im_A\psi_{k,2}^A(\eta')}\right)\right], \tag{B16}$$

where $T \sim \mathcal{O}(1/(\tilde{a}m_A))$. This averaged quantity will only be non-zero if η coincides with a stationary point of the phases $\psi_{k,1}^A(\eta)$ or $\psi_{k,2}^A(\eta)$. Seeing as $\psi_{k,2}^A(\eta)$ has no stationary points for physical values of $\tilde{a}$, we only obtain contributions from terms involving $\psi_{k,1}^A(\eta)$. On making the stationary phase approximation we arrive at

$$\left\langle \frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\psi + 3\tilde{p}_\psi)\right\rangle = \sum_{A,B} \frac{4(4\pi)}{(2\pi)^3\tilde{a}^5(\eta)}\int dk\delta(\eta - \eta_k^A) w_k(\eta_k^A) k^2 \times 2\Re\left[\mathcal{B}_k^A \mathcal{B}_k^{B*} e^{im_A\psi_{k,1}^A(\eta_k^A) - im_B\psi_{k,1}^B(\eta_k^B) + i(s_k^A - s_k^B)\pi/4}\right] \times \Theta(\eta_k^A - \eta_k^B)\Theta(\eta_k^B - \eta_0), \tag{B17}$$

where $\mathcal{B}_k^A$ is as defined in (105). On using the fact that $\mathcal{B}_k^A\mathcal{B}_k^{B*} = \mathcal{B}_k^{A*}B_k^B$, this can then be written as

$$\left\langle \frac{1}{2f}\frac{df}{d\tilde{t}}(-\tilde{\rho}_\psi + 3\tilde{p}_\psi)\right\rangle = \sum_{A} \frac{4(4\pi)}{(2\pi)^3\tilde{a}^5(\eta)}\int dk\delta(\eta - \eta_k^A) w_k(\eta_k^A) k^2 |\mathcal{B}_k^A|^2 + \sum_{A,B>A} \frac{4(4\pi)}{(2\pi)^3\tilde{a}^5(\eta)}\int dk\delta(\eta - \eta_k^A) w_k(\eta_k^A) k^2 \mathcal{B}_k^A\mathcal{B}_k^{B*} \times 2\cos\left(m_A\psi_{k,1}^A(\eta_k^A) - m_B\psi_{k,1}^B(\eta_k^B) + (s_k^A - s_k^B)\pi/4\right)\Theta(\eta_k^B - \eta_0). \tag{B18}$$

In the above expression we have arranged that $m_B > m_A$ for $B > A$, meaning that $\Theta(\eta_k^A - \eta_k^B) = 1$ only for $B > A$. Next we re-write the delta function in η as a delta function in k. The relation is as given in (81), but now with $\psi_{k,1}^{A\prime\prime}(\eta_k^A)$ as given in (B14) and μ_A to leading order in α^A given as

$$\mu_A \simeq \frac{m_A}{2}\left(1 - \frac{4m_\psi^2}{m_{\hat{A}}^2}\right)^{1/2}. \tag{B19}$$

Assuming that the non-diagonal terms in the second line of (B18) average to zero, the diagonal terms give rise to (107).

---

[1] G. Hinshaw *et al.*, The Astrophysical Journal Supplement **208**, 19 (2013).
[2] P. Ade *et al.* (BICEP2, Planck), Phys.Rev.Lett. **114**, 101301 (2015), arXiv:1502.00612 [astro-ph.CO].
[3] P. Ade *et al.* (Planck), (2015), arXiv:1502.01589 [astro-ph.CO].
[4] P. Ade *et al.* (Planck), (2015), arXiv:1502.02114 [astro-ph.CO].
[5] E. J. Copeland, R. Easther, and D. Wands, Physical Review D **56**, 874 (1997).
[6] D. I. Kaiser and E. I. Sfakianakis, arXiv.org (2013), 1304.0363v2.
[7] A. A. Starobinsky, Physics Letters B **91**, 99 (1980).
[8] F. Bezrukov and M. Shaposhnikov, Physics Letters B **659**, 703 (2008).
[9] R. Kallosh, A. Linde, and D. Roest, JHEP **1311**, 198 (2013), arXiv:1311.0472 [hep-th]; (2013), arXiv:1310.3950 [hep-th]; R. Kallosh and A. Linde, JCAP **1310**, 033 (2013), arXiv:1307.7938; JCAP **1307**, 002 (2013), arXiv:1306.5220 [hep-th]; JCAP **1306**, 028 (2013), arXiv:1306.3214 [hep-th]; JCAP **1306**, 027 (2013), arXiv:1306.3211 [hep-th].
[10] D. I. Kaiser, E. A. Mazenc, and E. I. Sfakianakis, Physical Review D **87**, 064004 (2013).
[11] J. White, M. Minamitsuji, and M. Sasaki, Journal of Cosmology and Astroparticle Physics **07**, 039 (2012).
[12] R. N. Greenwood, D. I. Kaiser, and E. I. Sfakianakis, Phys.Rev. **D87**, 064021 (2013), arXiv:1210.8190 [hep-ph].
[13] J. White, M. Minamitsuji, and M. Sasaki, Journal of Cosmology and Astroparticle Physics **09**, 015 (2013).
[14] R. Kallosh and A. Linde, (2013), arXiv:1309.2015 [hep-th].

[15] K. Schutz, E. I. Sfakianakis, and D. I. Kaiser, Phys.Rev. **D89**, 064044 (2014), arXiv:1310.8285 [astro-ph.CO].
[16] L. Dai, M. Kamionkowski, and J. Wang, Phys.Rev.Lett. **113**, 041302 (2014), arXiv:1404.6704 [astro-ph.CO].
[17] M. A. Amin, M. P. Hertzberg, D. I. Kaiser, and J. Karouby, Int.J.Mod.Phys. **D24**, 1530003 (2015), arXiv:1410.3808 [hep-ph].
[18] J. Martin, C. Ringeval, and V. Vennin, (2014), arXiv:1410.7958 [astro-ph.CO].
[19] J.-O. Gong, S. Pi, and G. Leung, (2015), arXiv:1501.03604 [hep-ph].
[20] Y. Watanabe, Phys.Rev. **D85**, 103505 (2012), arXiv:1110.2462 [astro-ph.CO].
[21] G. Leung, E. R. Tarrant, C. T. Byrnes, and E. J. Copeland, JCAP **1209**, 008 (2012), arXiv:1206.5196 [astro-ph.CO].
[22] J. Meyers and E. R. M. Tarrant, Phys.Rev. **D89**, 063535 (2014), arXiv:1311.3972 [astro-ph.CO].
[23] J. Elliston, S. Orani, and D. J. Mulryne, Phys.Rev. **D89**, 103532 (2014), arXiv:1402.4800 [astro-ph.CO].
[24] A. Vilenkin, Phys.Rev. **D32**, 2511 (1985).
[25] B. A. Bassett and S. Liberati, Phys.Rev. **D58**, 021302 (1998), arXiv:hep-ph/9709417 [hep-ph].
[26] M. B. Mijic, M. S. Morris, and W.-M. Suen, Phys.Rev. **D34**, 2934 (1986).
[27] Y. Watanabe and E. Komatsu, Phys.Rev. **D75**, 061301 (2007), arXiv:gr-qc/0612120 [gr-qc].
[28] T. Faulkner, M. Tegmark, E. F. Bunn, and Y. Mao, Phys.Rev. **D76**, 063505 (2007), arXiv:astro-ph/0612569 [astro-ph].
[29] Y. Watanabe and E. Komatsu, Phys.Rev. **D77**, 043514 (2008), arXiv:0711.3442 [hep-th].
[30] D. Gorbunov and A. Panin, Phys.Lett. **B700**, 157 (2011), arXiv:1009.2448 [hep-ph].
[31] Y. Watanabe, Phys.Rev. **D83**, 043511 (2011), arXiv:1011.3348 [hep-th].
[32] E. Arbuzova, A. Dolgov, and L. Reverberi, JCAP **1202**, 049 (2012), arXiv:1112.4995 [gr-qc].
[33] Y. Watanabe and J. Yokoyama, Phys.Rev. **D87**, 103524 (2013), arXiv:1303.5191 [hep-th].
[34] Y. Ema, R. Jinno, K. Mukaida, and K. Nakayama, (2015), arXiv:1502.02475 [hep-ph].
[35] D. I. Kaiser, Phys.Rev. **D81**, 084044 (2010).
[36] G. Dvali, A. Gruzinov, and M. Zaldarriaga, Phys.Rev. **D69**, 023505 (2004), arXiv:astro-ph/0303591 [astro-ph].
[37] L. Kofman, (2003), arXiv:astro-ph/0303614 [astro-ph].
[38] M. Zaldarriaga, Phys.Rev. **D69**, 043508 (2004), arXiv:astro-ph/0306006 [astro-ph].
[39] T. Suyama and M. Yamaguchi, Phys.Rev. **D77**, 023505 (2008), arXiv:0709.2545 [astro-ph].
[40] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys.Rev. **D56**, 3258 (1997), arXiv:hep-ph/9704452 [hep-ph].
[41] R. Allahverdi, R. Brandenberger, F.-Y. Cyr-Racine, and A. Mazumdar, Ann.Rev.Nucl.Part.Sci. **60**, 27 (2010), arXiv:1001.2600 [hep-th].
[42] A. Dolgov and S. Hansen, Nucl.Phys. **B548**, 408 (1999), arXiv:hep-ph/9810428 [hep-ph].
[43] L. Parker, Phys.Rev. **183**, 1057 (1969).
[44] L. Parker, Phys.Rev. **D3**, 346 (1971).
[45] Y. Zeldovich and A. A. Starobinsky, Sov.Phys.JETP **34**, 1159 (1972).
[46] A. A. Starobinsky, in *Quantum Gravity*, edited by M. A. Markov and P. C. West (Plenum Press, New York, 1984) pp. 103–128.
[47] J. Braden, L. Kofman, and N. Barnaby, JCAP **1007**, 016 (2010), arXiv:1005.2196 [hep-th].
[48] B. S. DeWitt, Phys.Rept. **19**, 295 (1975).
[49] M. Peloso and L. Sorbo, JHEP **0005**, 016 (2000), arXiv:hep-ph/0003045 [hep-ph].
[50] H. P. Nilles, M. Peloso, and L. Sorbo, JHEP **0104**, 004 (2001), arXiv:hep-th/0103202 [hep-th].
[51] D. J. Chung, L. L. Everett, H. Yoo, and P. Zhou, Phys.Lett. **B712**, 147 (2012), arXiv:1109.2524 [astro-ph.CO].
[52] K.-i. Maeda, Phys.Rev. **D39**, 3159 (1989).
[53] Y. Fujii and K.-i. Maeda, *The Scalar-Tensor Theory of Gravitation* (Cambridge University Press, Cambridge, 2003).
[54] T. Koivisto, Class.Quant.Grav. **23**, 4289 (2006), arXiv:gr-qc/0505128 [gr-qc].